Derivation of a General Formula of Aesthetic Curves

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Abstract

The curve is the most basic design element to determine shapes and silhouettes of industrial products and works for shape designers and it is inevitable for them to make it aesthetic and attractive to improve the total quality of the shape design. Harada et al. insist that natural aesthetic curves like birds' eggs and butterflies' wings as well as artificial ones like Japanese swords and key lines of automobiles have such a property that their logarithmic curvature histograms(LCHs) can be approximated by straight lines and there is a strong correlation between the slopes of the lines and the impressions of the curves.

In this paper, we define the LCH analytically with the aim of approximating it by a straight line and propose new expressions to represent an aesthetic curve whose LCH is given exactly by a straight line. Furthermore we derive a general formula of aesthetic curves that describes the relationship between their radiuses of curvature and lengths.

Keywords: aesthetic curve, logarithmic curvature histogram, LCH line, extended clothoid curve

1 Introduction

For industrial designers, the curve is one of the most basic design parts that determines shapes and silhouettes of their products and works. It is necessary to make it aesthetically beautiful and attractive to improve the quality of the industrial design. Harada[1] pointed out that the logarithmic curvature histograms(LCHs) of aesthetically beautiful curves of nature such as birds' eggs and wings of butterflies as well as those of the artifact such as Japanese swords and key lines of automobiles can be approximated by straight lines. Furthermore, the slopes of the approximated lines are strongly related to the impressions of the curves. However their definition of the LCH was not strictly defined mathematically and that is done procedurally and numerically.

On the other hand, although Nakano et al.[2] analytically defined the LCH, their definition does not directly give conditions where the LCH can be approximated by a straight line, or in case where the LCH can be approximated, it can not directly determine the slope of the line. Moreover, if the shapes of the curves are obtained by their images, since only discretized data are available, the LCH graph calculated based on their definition is translated in the vertical direction from the LCH graph calculated by Harada's method as explained Section 3.

Therefore, in this paper we propose a method to define the LCH analytically with the intension of approximating it by a straight line and formulate the curve whose LCH is strictly given by a straight line with an arbitrary slope. Furthermore, we derive a general formula of aesthetic curves from the relationship between the arc length and the radius of curvature of the curve.

2 Quantification of beauty of curves

Here we describe the definition of the LCH given by Harada[1] and verify the validity of his method to quantify the beauty of the curve. They assumed that the subjects of his method were 1) planar curve and 2) the curve whose curvature varies monotonically. Therefore they did not deal with the curve whose curvature is constant such as the straight line and the circle[1].

2.1 Logarithmic curvature histogram

At first, we will make a LCH according to the method proposed by Harada[1]. An image such as shown in Fig.1(a) is binarized and the points on the sword curve are sampled discretely. These points are approximated by a B-spline curve as shown in Fig.1(b) and the radius of curvature at an arbitrary position on the curve is estimated.

The total length of the curve is denoted by S_{all} and the radius of curvature at the sampling point a_i is ρ_i. The sampling points $(a_1, a_2, \cdots, a_n)$ are extracted by the same interval and the radius of curvature data $(\rho_1, \rho_2, \cdots, \rho_n)$ are obtained by calculating the radius of curvature at each a_i by using the approximated B-spline curve.

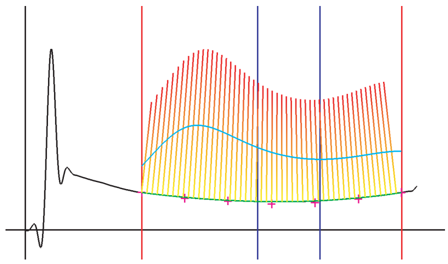
For example, if the total length of the curve is 1000mm and the sampling interval is 1mm, the number of the sampling points is 1001. Then the radius of curvature interval $\bar{\rho}_j$ is given by, for instance, subdividing by 100 the interval $[-3, 2]$, the logarithm of $[0.001, 100]$. The number N_j is calculated by counting ρ_i/S_{all} included in the interval $\hat{\rho}_j$ and the partial length s_j(=the interval of the sampling points$\times N_j$) is obtained. Furthermore the length frequency $\bar{s}_j$ is determined as the logarithm of the ratio of s_j to $S_{all}(\bar{s}_j = log_{10}(s_j/S_{all})$. By taking $\bar{s}_j$ in the horizontal direction and $\bar{s}_j$ in the vertical direction, the LCH is drawn as shown in Fig.1(c). In this paper, if the graph of the LCH can be approximated by a straight line, we call it the logarithmic curvature histogram line.

[1]Although the line and the circle are beautiful, their beauty originates from their simplicity and we do not deal with these curves.

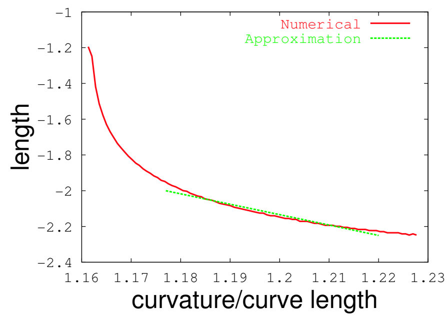
2.2 Impression of the curve

The LCHs of many natural and artificial curves can be approximated by straight lines. Harada[1] insisted that the impressions of the curves and their slope of the LCH lines are strongly related and the relationships between the impressions and the slope can be summarized as described in the table 1. His statements made the characters of the beautiful curves quantitatively clearer than the quantization criteria previously proposed by Higashi et al.[3] that evaluates the monotonic variation of curvature.

(a) Japanese sword



(b) Approximation by a B-spline curve



(c) LCH and its approximation line

Figure 1: Generation of LCH

rhythm	α	elem. func.	impressions
simple	$-$	sin, cubic cur.	sharp, strong
	0	(not found)	stable
	$+$	parabola, log, log. spiral	gathering centripetal
complex	$+ \to -$	sin	diverge to converge
	$- \to +$	(not found)	converge to diverge

Table 1: LCH lines' slopes α and their impressions

3 Analytically defined logarithmic curvature histogram

As explained in the previous section, Harada's definition of the LCH is not analytical and for example, the length frequency of the curve can not be evaluated at a certain position on the curve or for a given value of the radius of curvature. In this section, we think about how to define the LCH analytically.

Nakano et al.[2] showed that for a given curve $\boldsymbol{C}(t) = (x(t), y(t))$, the derivative of the arc length s with respect to the logarithm of the radius of curvature $R = \log\rho$ is given by

$$\frac{ds}{dR} = \frac{(x'y'' - x''y')(x'^2 + y'^2)^{\frac{3}{2}}}{3(x'x'' + y'y'')(x'y'' - x''y') - (x'^2 + y'^2)(x'y''' - x'''y')} \quad (1)$$

where $'$ denotes the derivative with respect to the parameter t. The LCH is mathematically equivalent to the graph whose horizontal and vertical coordinates represent R and $\log(ds/dR)$, respectively.

Equation (1) is enough to analytically define the LCH. However it does not give any concrete conditions for the parameter ranges where the LCH can be approximated by a straight line or determine the slope of the approximated line. Furthermore, for a curve whose shape is obtained from its image data, only discrete data are available and the partial arc length s_j must be a finite value to calculate the length frequency. As a result, the LCH graph is translated in the horizontal direction from that obtained by Eq.(1) as explained below.

Hence we think about a transformation of the left side of Eq.(1). Since the slope of the LCH graph is expressed by $\log(ds/d(\log\rho))$ and both of s and ρ are functions of the parameter t,

$$\begin{aligned} \log\frac{ds}{d(\log\rho)} &= \log\frac{\frac{ds}{dt}}{\frac{d(\log\rho)}{dt}} = \log(\rho\frac{\frac{ds}{dt}}{\frac{d\rho}{dt}}) \\ &= \log\rho + \log s_d - \log\frac{d\rho}{dt} \end{aligned} \quad (2)$$

where $s_d = ds/dt$. Equation (2) is defined by the radius of curvature and its derivative and describe the relationship between the radius of curvature and the derivative of the arc length more explicitly than Eq.(2).

In the next subsection, we make analytically clear the relationship between the logarithm of the finite change of the arc length $\Delta s = (\rho ds/d\rho)\Delta\log\rho)$ and $\log\rho$. We use the parabola as an analysis example to make the discussion more understandable.

3.1 Parabola

We assume that a parabola is given by $\boldsymbol{C}(t = x) = (x, ax^2)$ by letting $t = x$ where a is a positive constant.

If the small change $\Delta\log\rho$ of the logarithm $\log\rho$ of the radius of curvature ρ is a constant c, then

$$\Delta s = \frac{ds}{d(\log\rho)}\Delta\log\rho = \frac{ds}{d(\log\rho)}c \quad (3)$$

By taking the logarithm of both sides of the above equation and by using Eq.(2),

$$\log\Delta s = \log\rho + \log s_d - \log\frac{d\rho}{dx} + \log c \quad (4)$$

For the parabola, ρ and s_d are given by the following expressions:

$$\rho = \frac{(1 + 4a^2x^2)^{\frac{3}{2}}}{2a}, \quad s_d = (1 + 4a^2x^2)^{\frac{1}{2}} \quad (5)$$

Therefore, by considering $d\rho/dx = 6axs_d$,

$$\log\Delta s = \log\rho - \log 6ax + \log c. \quad (6)$$

From Eq.(5), x can be expressed by ρ as follows:

$$x = \frac{1}{2a}\{(2a\rho)^{\frac{2}{3}} - 1\}^{\frac{1}{2}}. \quad (7)$$

Since the above expression can be approximated by $x \approx (2a\rho)^{\frac{1}{3}}/2a$ when $(2a\rho)^{2/3} \gg 1$, Eq.(6) becomes

$$\log\Delta s = \frac{2}{3}\log\rho + C \quad (8)$$

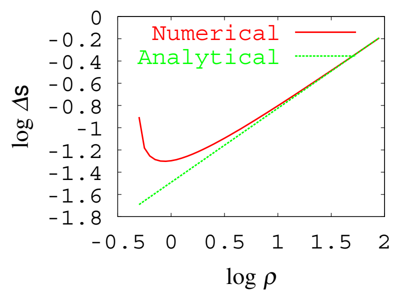
where $C = -\log a/3 - \log 2/3 - \log 3 + \log c$. Hence the slope of the LCH line is equal to $2/3$.

Figure 2(a) shows the LCH produced by the numerical method mentioned in subsection 2.1 and the line analytically obtained by Eq.(8). From this figure, if x is larger than $x \approx 0.778(\log_{10}\rho = 0.5)$, we can say that Eq.(8) approximates the LCH graph very well.

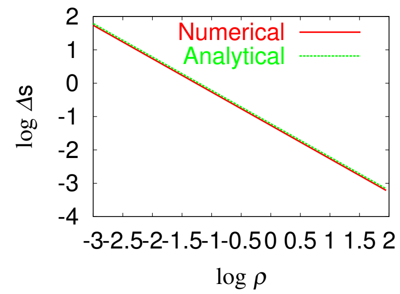
The value of the slope is equal to that mentioned by Harada[1] and we make clear the condition $(2a\rho)^{2/3} \gg 1$ that is necessary to approximate the graph by a straight line very well. That the LCH graph is given by a straight line means $\Delta s/\rho^{\alpha} = const$ form $\log \Delta s = \alpha \log \rho + C$ or the α-th power of the radius of curvature ρ is proportional to small change of the arc length Δs.

Based on the above discussion, we find out that the LCH graph defined by Eq.(2) is translated in the vertical direction from that obtained by using the length frequency by $\log c$.

We perform the similar analysis for the clothoid curve and its result is shown in Fig.2(b). As known from this figure, the slope of the LCH line of the clothoid curve is equal to -1 and the graph is strictly expressed by a straight line for an arbitrary parameter value.



(a) Parabola



(b) Clothoid curve

Figure 2: LCHs and LCH lines

3.2 Curve with an arbitrary LCH line slope

For the design of curves, it is desirable to represent a curve whose LCH line can have an arbitrarily valued slope. Since the radius of curvature of the clothoid curve can be formulated by a simple expression, we consider extensions of the clothoid curve to make them have an arbitrary slope for their LCH lines.

Here we will apply the fine tuning method[4] to the clothoid curve and extend its representation. The fine tuning method can scale curvature at a point on curves and surfaces to an arbitrary value. In a curve case, for a given curve $\boldsymbol{C}(t)$, by using a scalar function $g(t) > 0$ and define a new curve as follows:

$$\boldsymbol{C}'(t) = \boldsymbol{P}_0 + \int_0^t g(t) \frac{d\boldsymbol{C}(t)}{dt} dt \tag{9}$$

Namely differentiate the original curve, scale the first derivative by multiplying a scale function and change the value of curvature arbitrarily. The clothoid curve applied by the fine tuning(Fine Tuned Clothoid : FTC) is defined by the following expression in the complex plane:

$$\boldsymbol{C}(t) = \int_0^t g(t) e^{iat^2} dt \tag{10}$$

where i is the imaginary unit, a is a constant and $g(t)$ is a scale function whose value is always positive.

By using the radius of curvature ρ_c of the clothoid curve, we define $g(t) = (1/2at)^{\beta}$ If we assume β can be positive or negative values, $g(t)$ is equivalent to be the $-\beta$-th power of t except for the constant coefficient. The analysis results yields

$$\log \Delta s = \frac{\beta - 1}{\beta + 1} \log \rho + C \tag{11}$$

where $C = -\log(\beta+1) - \log 2 - \log a + \log c$. Hence the LCH graph is given by a straight line whose slope is $(\beta-1)/(\beta+1)$ and the slope α can be an arbitrary value except for 1[2]. Figure 3 shows several FTC curves whose LCH lines' slopes are given by α. The curve whose α is equal to -1 is a clothoid curve.

The FTC curve which has 1 for its LCH line slope can be obtained with $g(t) = c_0 t e^{c_1 t^2}$ by solving a differential equation $\Delta s/\rho = const$ where c_0 and c_1 are constants.

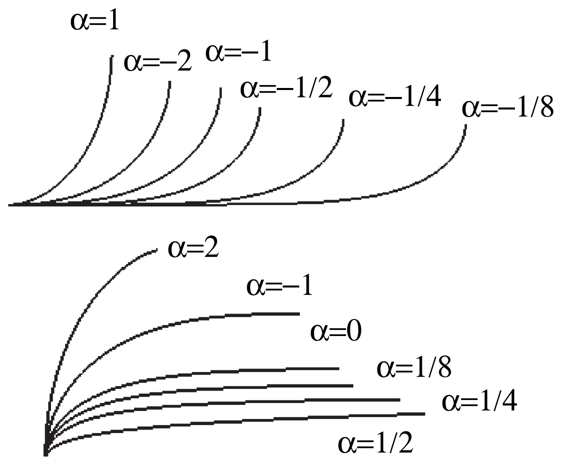


Figure 3: Curves whose LCH graphs are given by α-sloped straight lines

4 A general formula of aesthetic curves

In this section, we derive a formula of the curve whose LCH is given by a straight line. The curve obtained here can represent aesthetic curves and we call it a general formula of aesthetic curves

4.1 Derivation of a general formula

If we take the limit of $\Delta \log \rho$ to be 0, from the assumption that the LCH graph is given by a straight line, for the left side of Eq.(2) there is a constant α and

$$\log(\rho \frac{ds}{d\rho}) = \alpha \log \rho + C \tag{12}$$

where C is a constant. We call this the basic equation of aesthetic curves. By transforming Eq.(eq:aesthetic0), we obtain

$$\frac{1}{\rho^{\alpha-1}} \frac{ds}{d\rho} = e^C = C_0 \tag{13}$$

Hence

$$\frac{ds}{d\rho} = C_0 \rho^{\alpha-1} \tag{14}$$

If $\alpha \neq 0$,

$$s = \frac{C_0}{\alpha} \rho^\alpha + C_1 \tag{15}$$

In the above equation, C_1 is an integral constant. Therefore

$$\rho^\alpha = C_2 s + C_3 \tag{16}$$

where $C_2 = \alpha/C_0$ and $C_3 = -(C_1\alpha)/C_0$. Here we rename C_2 and C_3 to c_0 and c_1, respectively. Then

$$\rho^\alpha = c_0 s + c_1 \tag{17}$$

The above equation indicates that the α-th power of the radius of curvature ρ is given by a linear function of the arc length s.[3] The above equation is called a general formula of aesthetic curves in this paper.[4]

The logarithmic(equiangular) spiral and clothoid curve are regarded as two typical beautiful curves. One of the principal characters of the logarithmic spiral that its radius of curvature and arc length are proportional is well known and it means that the logarithmic spiral satisfies Eq.(17) and its α is equal to 1. On the other hand the main property of the clothoid curve is that its radius of curvature is in inverse proportion to its arc length. Eq.(17) is satisfied for the clothoid curve if α is given by -1.

In summary, the general formula of aesthetic curves expressed by Eq.(17) includes the most typical beautiful curves such as the logarithmic spiral and the clothoid curve.

[2] If β is equal to -1, the curve becomes a circle

[3] Note that the local property that the α-th power of the radius of curvature ρ is proportional to the small change of the arc length Δs is satisfied globally for the whole curve.

[4] If we do not care about the derivation of the general aesthetic formula, note that when $c_0 = 0$, it can represent lines and circles whose radius of curvature is constant.

4.2 Parametric expression of the general aesthetic curves

In this subsection, we find a parametric expression of the general formula of aesthetic curves given by Eq.(17).

We assume that a curve $\boldsymbol{C}(s)$ satisfies Eq.(17). Then

$$\rho(s) \quad = \quad (c_0 s + c_1)^{\frac{1}{\alpha}} \tag{18}$$

As s is the arc length, $|s_d| = 1$(refer to, for example, [5]) and there exists $\theta(s)$ satisfying the following two equations:

$$\frac{dx}{ds} = \cos\theta, \quad \frac{dy}{ds} = \sin\theta \tag{19}$$

Since $\rho(s) = 1/(d\theta/ds)$,

$$\frac{d\theta}{ds} = (c_0 s + c_1)^{\alpha} \tag{20}$$

Hence

$$\theta = \frac{(c_0 s + c_1)^{\alpha+1}}{(\alpha+1)c_0} + c_2 \tag{21}$$

If the start point of the curve is given by $\boldsymbol{P}_0 = \boldsymbol{C}(0)$,

$$\boldsymbol{C}(s) = \boldsymbol{P}_0 + e^{ic_2} \int_0^s e^{i\frac{(c_0 u + c_1)^{\alpha+1}}{(\alpha+1)c_0}} du \tag{22}$$

The above expression can be regarded as an extension of the clothoid curve whose power of e in its definition is changed from 2 to $\alpha + 1$ and its LCH line's slope can be specified to be equal to any value except for 0.

5 Conclusion

Harada's work is very suggestive to analyze the characters of aesthetic curves. In this paper, based on his work we have defined the LCH analytically with the purpose of approximating it by a straight line and formulated the curve whose LCH graph is strictly expressed by a straight line. Furthermore we have found out the relationship between the radius of curvature and the arc length of the curve whose LCH graph is given by a straight line and proposed it as a general formula of aesthetic curves. We have shown that the logarithmic spiral and the clothoid curve that are the two most typical aesthetic curves satisfy the general formula of aesthetic curves newly proposed in this paper.

For future work, we are planning an automatic classification of curves: 1) determine the rhythm to be simple(monotonic) or complex(consisting of plural rhythms), 2) calculate the slope of the line approximating the LCH graph. We think there are a lot of possibilities to use the general aesthetic formula to many applications in the fields of computer aided geometric design. For example, we may be able to apply the formula to deform curves to change their impressions, say, from shape to stable. Another example is smoothing for reverse engineering. Even if only noisy data of curves are available, we may be able to use the formula as a kind of rulers to smooth out the data and yield aesthetically high quality curves. We will develop a CAD system using the formula.

Acknowledgments

The authors thank Yasuomi Murota for his graduation thesis work. A part of this research is supported by the Grant-in-Aid Scientific Research(C) (15560117) from 2003 to 2004.

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