Abstract

We describe a simple geometric construction for Bézier conics and quadrics, based on a tool called Weighted Radial Displacement (WRD). The shape of a rational Bézier curve or surface is modified via WRD by choosing an arbitrary point $O$ and displacing the control points along radial directions through $O$, changing simultaneously the weights. To construct a conic through $O$, take an arbitrary segment representing a curve of degree $n = 1$, degree raise it to $n = 2$ and apply a WRD. Analogously, if a degree-elevated triangle is modified using a WRD, we get a quadric through $O$.

Any quadratic Bézier patch on a nondegenerate quadric, which defines a stereographic projection, can be obtained through this method. We present a practical algorithm to detect such quadratic Bézier patches lying on quadrics. Bézier patches on degenerate quadrics are also derived via a WRD. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

The rational Bernstein–Bézier scheme has become a de facto standard for the representation of curves and surfaces in CAGD (Farin, 1997), primarily because it encompasses under a unified mathematical model both freeform and classical analytical...
shapes. Among the latter, algebraic curves and surfaces of degree \( n = 2 \) (conics and quadrics) traditionally play an important role in architecture and mechanical engineering. Thus, the problem of representing patches on a quadric as Bézier surfaces has been addressed in a number of articles. The simplest representation is a quadratic triangular patch (Lodha and Warren, 1990; Boehm and Hansford, 1991; Teller and Sequin, 1991), yet a rectangular tensor-product representation is also possible (Boehm, 1993; Dietz et al., 1993, 1995).

Another question to solve is how to detect whether a given quadratic triangular patch lies on a quadric or not. In general, such a patch lies on an implicit surface of degree 4, called Steiner surface (Piegl, 1985; Sederberg and Anderson, 1985). Boehm and Hansford, 1991 gave three conditions for a quadric patch to lie on a quadric, namely that the three boundary curves:

1. Meet in one point \( O \).
2. Have coplanar tangents at \( O \).
3. Assume the parameter value \( t = \infty \) at \( O \).

Nevertheless, Boehm and Prautzsch (1994), in Remark 6 of Chapter 23, do not mention condition (3). Analogously, Farin listed these three conditions in his book on NURBS (Farin, 1995), but dropped the last one in the second, third and fourth edition (Farin, 1997) of his book on CAGD. This is hence a somewhat obscure point in the CAGD literature.

In addition, as one of the referees pointed out, the case where a boundary is a straight line should also be properly considered.

The conditions above are geometric and simple to enunciate. However, as Farin (1995) points out, they cannot be checked in a numerically reliable way in many cases. That is why Albrecht (1998a, 1998b) recently developed an alternative method, utilizing results due to Degen (1996) on Veronese surfaces in 5-dimensional projective space.

In this article, in addition to confirming that the third condition is actually unnecessary, we show that most results regarding Bézier conics and quadrics take a very simple form in terms of a geometric construction, which we called Weighted Radial Displacement (WRD for short). Given a rational Bézier curve or surface, we modify it via WRD by choosing an arbitrary point \( O \), and then displacing each control point along the corresponding radial direction through \( O \). Simultaneously, we change the weight associated to the control point, keeping the product weight by distance to \( O \) constant. If we take a segment, (a linear Bézier curve of degree one) degree raise it up to degree two, and modify it via WRD, a conic through \( O \) is obtained. Analogously, from a degree-elevated triangle we generate a quadric through \( O \). The WRD concept also yields simple conditions to solve the “quadric or not” problem, without resorting to exotic surfaces in higher-dimensional spaces.

The paper is arranged as follows. In Section 2, the chief properties of the WRD are reviewed. In Section 3, we discuss how to generate conics via WRD. The WRD construction also furnishes simple conditions to determine if a given point lies on the conic defined by a Bézier curve, as shown in Section 4. The construction of Bézier patches on quadrics via WRD is carried out in Section 5. Conditions for a quadratic Bézier patch to lie on a quadric are analysed in detail in Section 6. In Section 7 we develop a special construction for patches on degenerate quadrics (cones and cylinders) that is also based on a WRD. Finally, conclusions are summarized in Section 8.
2. Weighted Radial Displacement (WRD)

The Weighted Radial Displacement (WRD) constitutes a unified approach to NURBS shape modification, which builds on a perspective functional transformation of arbitrary origin (Sánchez-Reyes, 1997). In this chapter we briefly review this tool, focusing our attention on the particular Bézier case, as our final goal is the generation of Bézier conics and quadrics.

2.1. Defining a WRD

A planar rational Bézier curve $b(u)$, with control points $b_i = [x_i, y_i]^T$ and weights $w_i$, can be interpreted as the perspective projection onto the plane $z = 1$ of a polynomial Bézier curve in Euclidean 3D space. Control points of this nonrational curve have homogeneous coordinates:

$$[x_i w_i y_i w_i w_i]^T = [p_i w_i]^T, \quad p_i = w_i b_i.$$  \hspace{1cm} (1)

Hence, a degree-$n$ rational Bézier curve $b(u), u \in [0, 1]$, can be written as:

$$b(u) = \frac{p(u)}{w(u)}, \quad p(u) = \sum_{i=0}^{n} p_i B_i^n(u), \quad w(u) = \sum_{i=0}^{n} w_i B_i^n(u),$$  \hspace{1cm} (2)

where $B_i^n(u)$ denotes the Bernstein polynomials of degree $n$. The denominator $w(u)$ will be referred to as the weight function.

We choose now an arbitrary point $O$ and, without lack of generalization, we express the curve taking $O$ as the origin of coordinates, which greatly simplifies the resulting formulas. Note that changing the origin modifies the coordinates of points $b_i$ by adding a common vector, whereas this does not apply to the homogeneous coordinates $p_i$. If we lift the 3D control points in the $z$-direction by choosing arbitrary factors $\lambda_i$:

$$[p_i w_i]^T \rightarrow [p_i \lambda_i w_i]^T,$$  \hspace{1cm} (3)

this operation moves $b_i$ along a radial direction passing through $O$ and, simultaneously, changes the associated weight $w_i$:

$$b_i \rightarrow b_i^* = b_i/\lambda_i, \quad w_i \rightarrow w_i^* = \lambda_i w_i.$$  \hspace{1cm} (4)

We can utilize the control point as shape handle: we slide the control point along the ray $Ob_i$ until reaching the desired new location $b_i^*$, derive the factor $\lambda_i$, and compute the new weight $w_i^*$ (4).

The operation (3) modifies the weight function, but not the numerator $p(u)$ of expression (2). Hence the new curve $b^*(u)$ is given by:

$$b^*(u) = \frac{p(u)}{w^*(u)} = \frac{b(u)}{\lambda(u)}, \quad \lambda(u) = \frac{w^*(u)}{w(u)},$$

where $\lambda^{-1}(u)$ denotes a functional scale factor. Therefore, $b^*(u)$ is a perspective functional transformation (with centre $O$) of the original curve $b(u)$, that is, a point corresponding to
a fixed parameter \( u \) moves on a radial direction through \( O \). Clearly, the WRD defines a mapping \( b(u) \rightarrow b^*(u) \), and the inverse map is also a WRD of centre \( O \) (and factors \( \lambda_i^{-1} \)).

For simplicity, we have considered only curves. Nevertheless, these concepts extend to surfaces in a straightforward manner, because a WRD in essence reduces to changing the last homogeneous coordinate of the control points, as indicated by expression (3).

2.2. Properties and particular cases of the WRD

Two interesting cases arise for particular locations of the centre \( O \):

- \( O \) at infinity and converted to a direction: the points \( b_i \) move along this direction, the weights remain unchanged and the WRD reduces to the standard displacement of control points.
- \( O \) on the curve, that is, \( O = b(u_0) \) for a certain parameter value \( u_0 \): the position and tangent direction of the curve at \( O \) do not vary.

Other two remarkable particular cases are those corresponding to homogeneous linear transformations of the whole curve:

- If all points \( b_i \) are submitted to a scaling of centre \( O \), the whole curve is scaled.
- If all points \( b_i \) are projected onto a line, using \( O \) as centre of projection, we have a central projection of the curve onto the line. In the case of a quadratic curve \( b(u) \) and \( O \) on the conic defined by \( b(u) \), this is a stereographic projection.

A positive property of the WRD is its invariance with respect to rational linear (Möbius) reparameterizations. If a curve is first reparameterized, and second submitted to a WRD defined by a set of parameters \( \lambda_i \), the result is the same than if we apply first the WRD and then reparameterize. This property is demonstrated in (Sánchez-Reyes, 1997) for curves. It also applies to triangular surfaces, because the reparameterization of such surfaces (Joe and Wang, 1994) just multiplies the original weights by certain factors, without moving the control points.

3. WRD for constructing Bézier conics

3.1. The linear case

As an introduction, we discuss the WRD in the case of a linear curve. This investigation also furnishes a geometric look at the Möbius reparameterization of a rational Bézier curve.

Consider a segment \( c_0 c_1 \) represented as a polynomial Bézier curve \( c(u) \) of degree \( n = 1 \), that is, with control points \( c_0, c_1 \), and unit weights. Choose a point \( O \) and apply a WRD to \( c(u) \). As a result (Fig. 1), the control points move to new locations \( b_0, b_1 \) along radial lines through \( O \), and the original unit weights change to \( w_0, w_1 \), so that:

\[
b_0 = c_0 / w_0, \quad b_1 = c_1 / w_1.
\]  

Observe that \( c(u) \) and \( b(u) \) are related by a central projection (of centre \( O \)). The resulting rational curve \( b(u) \) does not trace out the segment \( b_0 b_1 \) in a linear fashion any more, unless \( w_0 = w_1 \) and in consequence the WRD degenerates to a scaling.

Given an arbitrary linear rational curve \( b(u) \), we could always find a degree-1 curve \( c(u) \) with unit weights such that \( c(u), b(u) \) are related by a WRD: simply choose an arbitrary
point \( O \) and take the points \( c_0, c_1 \) according to Eq. (5). Since \( c(u) \) is an affine image of the parametric domain \( u \in [0, 1] \), we can think of a rational linear mapping \( u \rightarrow b(u) \) as the inverse of a central projection.

This construction provides a geometric interpretation for a Möbius reparameterization \( v(u) \) of a rational Bézier curve. In the Bernstein basis, such a rational linear function \( v(u) \) is given by two weights \( w_0, w_1 \):

\[
v(u) = \frac{w_1 u}{w_0(1 - u) + w_1 u}.
\]

If the segments \( c_0c_1 \) and \( b_0b_1 \) are affine images of the domains \( u \in [0, 1] \) and \( v \in [0, 1] \), respectively, then the WRD \( c(u) \rightarrow b(u) \) maps affine images of \( u \) to affine images of \( v(u) \).

3.2. The canonical construction of conics

In this section we present the simplest WRD-based construction of Bézier conics, the so-called canonical construction.

Consider an arbitrary segment \( c_0c_2 \) (an affine image of the domain \( u \in [0, 1] \)), endowed with a linear parameterization:

\[
c(u) = c_0(1 - u) + c_2u. \tag{6}
\]

We apply now the following two-step procedure:

1. Degree-elevate to \( n = 2 \) the linear polynomial Bézier curve \( c(u) \) through standard degree elevation. We obtain a quadratic Bézier representation of \( c(u) \) with control points \( \{c_0, c_1 = (c_0 + c_2)/2, c_2\} \) that still parametrizes in a linear fashion \( c_0c_2 \).

2. Choose a centre \( O \) (not on the line \( c_0c_2 \)) and apply a WRD to this quadratic curve, by selecting the weights \( w_i \). In a coordinate system with origin \( O \), the resulting curve \( b(u) \) takes the form:

\[
b(u) = \frac{c(u)}{w(u)}, \quad w(u) = \sum_{i=0}^{2} w_i B_i^2(u). \tag{7}
\]
The curve \( b(u) \) is clearly a conic segment, because any rational quadratic curve is a conic. In a nondegenerate case, i.e., \( b(u) \) is not a straight line, this conic displays the following properties (Fig. 2(a)):

- It passes through \( O = b(\pm \infty) \).
- Its tangent line \( L_O \) at \( O \) is parallel to the segment \( c_0c_2 \).

To demonstrate the first property, observe that in the quotient (7) the numerator \( c(u) \) (6) is linear, whereas the denominator \( w(u) \) is quadratic. Hence, the limit at \( u = \pm \infty \) vanishes.

The second property is easily checked by introducing the change of variable \( u \rightarrow 1/s \) and computing the derivative at \( s = 0 \).

As we apply the WRD and move the control points along the radial lines \( Oc_i \), we generate a family of conics with a common point \( O \) and tangent there. This comes as no surprise, because a WRD with centre at a point on a curve yields a new curve that passes through \( O \) and keeps the tangent \( L_O \), as commented in Section 2.2.

It is worth studying the particular case where we choose a centre \( O \) at infinity, that is, \( O \) transforms to a direction. To fix ideas, assume that this direction is the y-axis (Fig. 2(b)). To express the curve \( b(u) \) we use now a system of coordinates \( (x, y) \) with arbitrary origin. The WRD degenerates to a parallel displacement of the control points along the y-axis, keeping the original unitary weights. This geometry gives rise to control points \( b_i = [x_i, y_i] \) with abscissas \( x_i \) regularly spaced. Hence we get a parabola of vertical axis, the nonparametric quadratic Bézier curve:

\[
y(u) = \sum_{i=0}^{2} y_i B_i^2(u).
\]

Another interesting case occurs when the WRD affects only the middle point, so that \( b_0 = c_0 \), \( b_2 = c_2 \), and we get a curve \( b(u) \) in standard form \( (w_0 = w_2 = 1) \). Then \( c_0c_2 \) becomes a chord of the conic, and the ray \( Oc_1 \) a diameter, that is, it contains the centre \( C \).
of the conic (Lee, 1987). This chord, parallel to $L_O$, is also parallel to the shoulder tangent (Farin, 1995). Fig. 3 illustrates this geometry.

The inverse map $b(u) \rightarrow c(u)$ is clearly the stereographic projection of centre $O$ from the conic segment onto the line $c_0c_2$. This correspondence between stereographic maps and rational quadratic Bézier curves was already explored by Teller and Sequin (1991), who observed that it provides a simple inversion procedure (see also (Wang and Joe, 1995)).

Lodha and Warren (1990) present a construction of Bézier conic segments, called projective functional representation, that is strongly connected to the WRD construction. The designer chooses a reference triangle $c_0c_2O$, identifies a focal vertex $O$, and selects the Bézier points $b_i$ on the rays $Oc_0$, $Oc_1$ and $Oc_2$. However, the curve $b(u)$ is computed in an intricate way. The system finds the linear perspective functional transformation $\tau$ such that these rays are mapped to parallel lines and applies $\tau$ to the control polygon to create the control polygon for a nonparametric Bézier curve. Finally such a curve is mapped back to the reference triangle via $\tau^{-1}$ to generate $b(u)$. Clearly, the WRD proposed is conceptually simpler and provide us with a deeper geometric look at the parameterization of conic segments in Bézier form.

### 3.3. Canonical construction associated with a given conic segment and centre

If we are given an arbitrary straight line segment (i.e., a degenerate conic), it always admits a trivial canonical construction, where only the first step (degree-elevation) is carried out. Assume now that we are given an arbitrary segment of a nondegenerate conic $\Gamma'$, of endpoints $b_0, b_2$, and a point $O$ on $\Gamma'$. Can we generate this segment via the canonical construction described in the preceding section? In this section we check that the answer is always affirmative.

Trace out the rays $Ob_0, Ob_2$, draw any line parallel to the tangent $L_O$ and compute the intersection points $c_0, c_2$ with the rays. This is the starting segment $c_0c_2$. First we degree elevate it, obtaining the midpoint $c_1$. Second, we apply a WRD, by sliding $c_0, c_2$ along the rays until they match the points $b_0, b_2$, and moving $c_1$ until we recapture the tangent line at one of the endpoints (for instance $b_0$). Clearly the original conic $\Gamma'$ is reproduced, because
we have constructed a conic that shares with \( \Gamma \) three points \((O, b_0, b_2)\) and two tangents (at \(O\) and \(b_0\)).

As a consequence, note that the midpoint \(c_1\) must lie precisely on the ray defined by \(O\) and the point \(b_1\) where the tangents at \(b_0, b_2\) meet.

3.4. Alternative representation of a linearly parameterized segment

In step (1) of the canonical construction we obtained an integral quadratic Bézier representation of a linearly parameterized segment via standard degree-elevation. As an alternative, we may utilize a generalized degree elevation (Denker and Herron, 1997).

Given a linearly parameterized segment \(d_0d_2\), we multiply and divide its Bézier form by an arbitrary linear function \(bw.\) and divide by the unity:

\[
d(u) = \frac{[(1-u)d_0 + ud_2]\widehat{w}(u)}{[(1-u) + u]w(u)}, \quad \widehat{w}(u) = \widehat{w}_0(1-u) + \widehat{w}_2u.
\]

(8)

Thus we generate a rational quadratic Bézier curve \(d(u)\) that traces out with linear precision (Farin and Jung, 1995) the segment \(d_0d_2\). Formally speaking, we have inserted a base point \(u_0\), the root of \(bw.\):

\[
u_0 = \frac{\widehat{w}_0}{\widehat{w}_0 - \widehat{w}_2}.
\]

(9)

In the Bernstein–Bézier form (2), the quadratic curve \(d(u)\), of control points \(d_i\) and weights \(w_i\), takes the form:

\[
d(u) = \frac{p(u)}{\widehat{w}(u)} = \frac{\widehat{w}_0 d_0(1-u)^2 + \widehat{w}_1 d_1 2(1-u)u + \widehat{w}_2 d_2 u^2}{\widehat{w}_0(1-u)^2 + \widehat{w}_1 2(1-u)u + \widehat{w}_2 u^2}.
\]

(10)

Expanding (8) and equating with expression (10), we derive the inner weight and control point:

\[
\hat{w}_1 = \frac{1}{2}(\hat{w}_0 + \hat{w}_2), \quad d_1 = (1-\alpha) d_0 + \alpha d_2, \quad \alpha = \frac{\hat{w}_0}{\hat{w}_0 + \hat{w}_2}
\]

(11)

We have a degree of freedom in this generalized degree raising, namely the base point \(u_0\), or the factor \(\alpha\) determining the position of \(d_1\). For \(u_0\) at infinity (\(\alpha = 1/2\)) we recapture the standard degree elevation. The apparent two degrees of freedom \(\hat{w}_0, \hat{w}_2\) reduce to one, because multiplying and dividing by a constant the quotient (8) leaves it unchanged.

The invariant of this construction is the cross ratio \(cr\) of the four collinear points \([d_0, d_1, d_2, d(u_0)]\), corresponding to the parameter values \([0, \alpha, 1, u_0]\). Introducing the values of \(u_0\) (9), and \(\alpha\) (11), we find that \(cr(d_0, d_1, d_2, d(u_0)) = -1\), using the definition of cross-ratio given in (Boehm and Prautzsch, 1994). Therefore, these four points are harmonic.

The numerator of \(d(u)\) in expression (8) is a certain parabola \(p(u)\):

\[
p(u) = p(u) = [(1-u)d_0 + ud_2]\widehat{w}(u),
\]

(12)

As \(u_0\) is a root of \(\hat{w}(u)\), from (12) we deduce the following properties:

1. \(p(u)\) passes through the origin of coordinates \(O = p(u_0)\).
(2) Its tangent line $L_O$ at $O$ meets the line $d_0d_2$ at $d(u_0)$:

$$d(u_0) = d_0(1 - u_0) + d_2u_0. \quad (13)$$

We could also express $p(u)$ as a Bézier curve of control points $p_i$:

$$p(u) = \sum_{i=0}^{2} p_i B_i^2(u).$$

Equating with (12) we derive the expressions for $p_i$:

$$p_0 = \tilde{w}_0d_0, \quad p_2 = \tilde{w}_2d_2, \quad 2p_1 = \alpha p_0 + \beta p_2, \quad \alpha = \beta^{-1} = \frac{\tilde{w}_2}{\tilde{w}_0}. \quad (14)$$

3.5. Extended construction of conic segments

The generalized degree-elevation discussed in the preceding Section allows a more flexible construction of conics, which will be referred to as the extended construction.

Let us apply an arbitrary WRD of centre $O$ (not on the line defined by $d_0d_2$) to the straight line $d(u)$ (8). As the WRD does not modify its numerator, the resulting conic segment $b(u)$ takes the form:

$$b(u) = \frac{p(u)}{w(u)}, \quad p(u) = [(1 - u)d_0 + ud_2]\tilde{w}(u). \quad (15)$$

where $w(u)$ indicates the new weight function. As $p(u)$ and $b(u)$ are related by a WRD of centre $O = p(u_0)$, in a nondegenerate case the curve $b(u)$ inherits the position and tangent direction of $p(u)$ at $O$ (Fig. 4(a)). In other words:

(1) $b(u)$ passes through $O = b(u_0)$;

(2) Its tangent $L_O$ meets the line $d_0d_2$ at the point $d(u_0)$.

Since the centre $O$ lies on the conic $b(u)$, the WRD $b(u) \mapsto d(u)$ is again a stereographic projection onto the line $d_0d_2$. It corresponds to a more general case, where the line of

![Fig. 4. Extended construction for conics via WRD.](image)
projection is not necessarily parallel to the tangent $L_O$ at the centre of projection $O$. The extra degree of freedom provided by the extended degree-elevation allows us to choose the slope of $L_O$.

The limit case of a centre $O$ at infinity is noteworthy. Without loss of generality, assume that $O$ is converted to a direction along the $y$-axis (Fig. 4(b)). The WRD degenerates to a parallel displacement of the control points along this axis, keeping the original weights $\hat{w}_i$. This geometry yields a rational nonparametric curve:

$$y(u) = \frac{1}{\hat{w}(u)} \sum_{i=0}^{2} \hat{w}_i y_i B_i^2(u),$$

where we use a system of coordinates $(x, y)$ with arbitrary origin to express the curve $b(u) = [u, y(u)]$. This is a hyperbola with a vertical asymptote cutting the line $d_0 d_2$ at $d(u_0)$, since the denominator $w(u)$ vanishes for $u = u_0$. Needless to say, if $a = 1/2$ in (11), the extended construction reduces to the canonical one and we obtain a parabola, which corresponds to a limit hyperbola with a vertical asymptote lying at $u_0 = \infty$.

In a degenerate case, such that $b(u)$ is a straight line (Fig. 5), the WRD corresponds to the central projection described in Section 3.1. Hence, the two properties above transform to:

1. $b(u)$ has real (minimum) degree one and inherits the base point $u_0$ from $d(u)$.
2. The line $L_O$ defined by $O$ and $b(u_0)$ meets the line $d_0 d_2$ at $d(u_0)$.

Expression (15) yields $0/0$ for the image $I = b(u_0)$ of the base point, owing to the common factor $\hat{w}(u)$ in the numerator and denominator. Needless to say, to compute $b(u_0)$ we must remove such a common factor.

In this case, where $b(u)$ is a straight line, the line $L_O$ could still be considered the tangent to $b(u)$ at $u_0$ if we interpret $b(u)$ as a limiting hyperbola that degenerates to its asymptotes. The first asymptote coincides with the line itself, and the second one is the line $L_O$. This hyperbola, which degenerates to $L_O$ at $u_0$, suffers a sudden change in its tangent at $b(u_0)$. 

Fig. 5. Particular case of a degenerate conic $b(u)$ (i.e., a straight line).
3.6. Correspondence between the canonical and extended constructions

Suppose we have obtained segment of a conic $\Gamma$ via an extended construction with centre $O$. According to Section 3.3, we could generate the same conic segment through the standard construction. Clearly, we get different Bézier representations of the same conic segment, corresponding to different parametrizations. The correspondence between such representations admits a geometric interpretation (Fig. 6). The segments $c_0c_2$ and $d_0d_2$, affine images of the parameter domains in the canonical and extended constructions, respectively, are related through a perspective projection of centre $O$, which, as explained in Section 3.1, can be interpreted as a Möbius reparameterization.

4. Point classification algorithm for conics in Bézier form

4.1. Equivalent conditions for a point $O$ to lie on a conic

The WRD construction leads to several ways for determining whether a point $O$ lies on a conic $\Gamma$. Let us formalize this result.

**Theorem 1.** Given a point $O$ and a Bézier curve $b(u)$ of control points $b_i$ that are not aligned and weights $w_i$, the following statements are equivalent:

(i) Expressing $b(u)$ in a coordinate system with origin $O$, its numerator $p(u)$ (2) factors as the product (12):

$$p(u) = [(1-u)d_0 + ud_2]\tilde{w}(u), \quad \tilde{w}(u) = \tilde{w}_0(1-u) + \tilde{w}_2u.$$

(ii) $b(u)$ admits an extended construction with centre $O$.

(iii) $O$ lies on the conic $\Gamma$ defined by $b(u)$. 

Fig. 6. Correspondence between the standard and extended construction for conics.
(iv) The coordinates \([αβ]\) of \(2p_1\) in the basis \(\{p_0, p_2\}\), where \(p_i = w_i b_i\), are such that:
\[
αβ = 1.
\]  

**Proof.** It suffices to prove the equivalence between (i) and (ii), (iii), (iv):

(i) \(|\Leftrightarrow|\) (ii) From expression (15) we have (ii) \(|\Rightarrow|\) (i). The implication (i) \(|\Rightarrow|\) (ii) is also clear: the construction has centre \(O\), starting segment \(d_0d_2\), and base point \(u_0\) indicated by (9).

(i) \(|\Leftrightarrow|\) (iii) The implication (i) \(|\Rightarrow|\) (iii) is straightforward, because \(u_0\) is a root of \(\hat{w}(u)\). On the other hand, if (iii) is satisfied, then both components of \(p(u)\) vanish simultaneously for a certain \(u_0\), which implies (i).

(i) \(|\Leftrightarrow|\) (iv) The implication (i) \(|\Rightarrow|\) (iv) is clear from Eq. (14). Regarding (iv) \(|\Rightarrow|\) (i), just choose an arbitrary nonzero \(b_w\) and derive from (14) the values that allow factoring \(p(u)\):
\[
\hat{w}_2 = α\hat{w}_0, \quad d_0 = \frac{p_0}{w_0}, \quad d_2 = \frac{p_2}{w_2}. \quad \Box
\]  

**Remark 1.** Condition (iv) provides a simple algorithm to determine whether a given point \(O\) lies on the conic \(Γ\) defined by \(b(u)\). If \(O\) has coordinates \([x, y]\) in a given coordinate system, then \([α β]\) are linear functions of \(x, y\) and, in consequence, the left side of Eq. (16) is quadratic in \(x, y\). Hence, the equality (16) corresponds to the implicit equation \(f(x, y) = 0\) of \(Γ\). However, the point here is not to derive a closed-form for the coefficients in \(f(x, y) = 0\), but to note that, since we employ the WRD deformation, the homogeneous points \(p_i = w_i b_i\) are readily available and hence so are their coordinates \([α β]\) in the basis \(\{p_0, p_2\}\). Such coordinates play a key role in other computations, as exemplified in the following remark.

**Remark 2.** If \(O\) lies on \(Γ\), combining (9), (13), (14) we find a very simple expression, in terms of \([α β]\), for the direction defined by the tangent line \(L_O\):
\[
βp_2 = αp_0.
\]  

**Remark 3.** Theorem 1 also solves the inversion problem: given a quadratic curve \(b(u)\) and a point \(q\) on \(b(u)\), find the parameter value \(u_q\) such \(q = b(u_q)\). Just choose an arbitrary point \(O\) on \(b(u)\) and compute the points \(d_0, d_2\) (17). As the segment \(d_0d_2\) is an affine image of the parametric domain \([0, 1]\), the intersection between the ray \(Oq\) and the line \(d_0d_2\) corresponds to the affine image of \(u_q\).

4.2. Case of a degenerate conic

Theorem 1 does not apply in case the given curve \(b(u)\) is a straight line. However, it does if we assume now that the given point \(O\) does not lie on the line and rewrite (iii) as:

(iii): \(b(u)\) contains a base point \(u_0\), so that it has minimum degree one.

We are dealing with a particular instance of an extended construction, where \(b(u) = d(u)\) and only the first step (the extended degree-elevation) is performed. Therefore, the original proofs for (i) \(|\Leftrightarrow|\) (ii) and (i) \(|\Leftrightarrow|\) (iv) hold. The implication (iii) \(|\Rightarrow|\) (i) is trivial.
On the other hand, if (i) is satisfied, then \( p(u) \) vanishes for a certain \( u_0 \). As \( b(u) \) does not pass through the origin \( O \), then for \( u_0 \) the numerator of \( b(u) \) must vanish too, which implies (iii). Finally, note that Eq. (18) still indicates the direction of the line \( L_O \), which was defined as \( Ob(u_0) \) in a degenerate case.

4.3. Case of a point at infinity

The point classification for the case of a point \( O \) at infinity requires a special treatment. Now one wants to determine if a given direction corresponds to an asymptote of a hyperbola \( b(u) \) (or an axis of a parabola). Without loss of generality, suppose that such a direction is the \( y \)-axis.

Condition (16) involves the coordinates of \( p_1 \) in the basis \( \{p_0, p_2\} \), that is, the solutions of the linear system:

\[
2p_1 = \alpha p_0 + \beta p_2.
\]

If the control points are \( b_i = [x_i, y_i]^T \) in an original system of coordinates \( (x, y) \), in a system with origin moved to \( O = [0, -y_0]^T \) they transform to \( [x_i, y_0 + y_i]^T \). Hence, the linear equation (19) can be written as:

\[
2w_1 \begin{bmatrix} x_1 \\ y_0 + y_1 \end{bmatrix} = \alpha w_0 \begin{bmatrix} x_0 \\ y_0 + y_0 \end{bmatrix} + \beta w_2 \begin{bmatrix} x_2 \\ y_0 + y_2 \end{bmatrix}.
\]

Dividing by \( y_0 \) the second line of the equation above, for \( y_0 \to \infty \) we obtain:

\[
2w_1 \begin{bmatrix} x_1 \\ 1 \end{bmatrix} = \alpha w_0 \begin{bmatrix} x_0 \\ 1 \end{bmatrix} + \beta w_2 \begin{bmatrix} x_2 \\ 1 \end{bmatrix}.
\]

We conclude that, for a point \( O \) at infinity, in condition (iv) of Theorem 1 we must employ points \( p_i = w_i x_i \), where \( x_i \) denotes the projection of \( b_i \) onto the line \( y = 1 \), instead of \( p_i = w_i b_i \). Analogously, in condition (i) we take the projection \( x(u) \) of \( b(u) \) onto \( y = 1 \), instead of \( b(u) \).

5. WRD for constructing triangular Bézier quadrics

Henceforth we adopt the following notation concerning triangular Bézier patches. The boldface symbol \( u \) designates the customary barycentric coordinates \( u, v, w \), where \( u + v + w = 1 \). We denote a generic point \( b_{ijk} \) by \( b_i \). We also use for the corners the abbreviations:

\[
A = (n, 0, 0), \quad B = (0, n, 0), \quad C = (0, 0, n),
\]

and, in the quadratic case, for the inner points:

\[
AB = (1, 1, 0), \quad BC = (0, 1, 1), \quad CA = (1, 0, 1).
\]
5.1. The linear case

As for curves, to begin we investigate the WRD in a linear case. Take an arbitrary triangle \( \Delta_c \) of vertices \( c_A, c_B, c_C \) endowed with a linear parameterization:

\[
c(u) = c_A u + c_B v + c_C w,
\]

and interpret it as a polynomial triangular Bézier patch of degree \( n = 1 \). Choose a point \( O \) and apply a WRD to \( c(u) \). As a result, the control points \( c_i \) move to new locations \( b_i \) along radial lines through \( O \), and the original unit weights change to values \( w_i \) such that:

\[
b_i = c_i / w_i.
\]

Therefore, \( c(u) \) and \( b(u) \) are related by a projection of centre \( O \) (see Fig. 7). In addition, given an arbitrary rational triangular surface \( b(u) \) of degree one, we could always find a polynomial patch \( c(u) \) such that \( c(u) \) and \( b(u) \) are related by a WRD: simply choose an arbitrary point \( O \) and take the points \( c_i \) indicated by the equality (21). As \( c(u) \) is an affine image of the barycentric domain \((u, v, w)\), we can always interpret a rational linear mapping \( u \mapsto b(u) \) as the inverse of a central projection.

Similarly to the curve case, this construction leads to a geometric interpretation of a Möbius reparameterization:

\[
v(u) = \frac{w_A A u + w_B B v + w_C C w}{w_A u + w_B v + w_C v},
\]

where \( A, B, C \) denote points in parameter space with barycentric coordinates \( A, B, C \). If the triangles \( \Delta_c \) and \( \Delta_d \) are affine images of the barycentric domains \( u \) and \( v \), respectively, then the WRD \( c(u) \mapsto b(u) \) maps affine images of \( u \) to affine images of \( v(u) \).

![Fig. 7. WRD for triangular surfaces: the linear case.](image-url)
5.2. The canonical construction of quadrics

In analogy to the case of conics, we adopt the term *canonical construction* to denote the simplest WRD-based construction of quadrics. Consider an arbitrary triangle $\Delta_c$ expressed in Bézier form $c(u)$ (20) and apply the following two-step procedure:

1. Degree-elevate to $n = 2$ the triangle $\Delta_c$ through standard degree elevation. We obtain three new control points, the midpoints of the edges of $\Delta_c$, corresponding to a quadratic Bézier representation of $c(u)$ that still parameterizes $\Delta_c$ in a linear fashion.

2. Choose a centre $O$ (not on the plane defined by $\Delta_c$) and apply a WRD to this quadratic surface. The resulting quadratic Bézier patch is:

$$ b(u) = \frac{c(u)}{w(u)} $$

where $w(u)$ denotes a quadratic weight function.

The question is whether $b(u)$ lies on certain quadric $Q$, and the answer is affirmative. The initial homogeneous triangle $[c(u)]$ lies on an affine 2D plane in $\mathbb{R}^4$. The WRD moves their control points along a fixed direction, as only the last homogeneous component is altered. Therefore, this occurs in a 3D hyperplane in $\mathbb{R}^4$. In fact, it is the nonparametric construction of a paraboloid as the graph of a quadratic function (Boehm and Hansford, 1991). The surface $b(u)$ stems from projecting this paraboloid onto the hyperplane $w = 1$, thereby lying on a quadric.

Observe that this 3D WRD reduces to a 2D WRD at each of the three boundary planes $[u = 0, v = 0, w = 0]$. Therefore all nondegenerate boundary conics of $b(u)$ display the following properties (Fig. 8):

1. They pass through $O$ for parameter values $\pm \infty$. 

![Fig. 8. Canonical construction for Bézier quadrics.](image)
(2) At \( O \) they are coplanar, since they have tangents parallel to the edges of \( \Delta_c \).

It is worth analysing the particular case of a centre \( O \) at infinity. The WRD degenerates to a parallel displacement of the control points, keeping the original unit weights. This construction yields the graph of a nonparametric function, in other words, the resulting patch lies on a paraboloid.

Similarly to the case of curves the WRD \( b(\mathbf{u}) \to d(\mathbf{u}) \) is the inverse of an stereographic projection from \( Q \) onto the projection plane \( P \) defined by the triangle \( \Delta_c \). This geometry allows a simple inversion procedure (Teller and Sequin, 1991).

5.3. Conditions for a patch to admit the canonical construction

In the 2D case we have seen that, given an arbitrary point \( O \) and conic segment, we can always obtain via the canonical construction a Bézier curve \( b(\mathbf{u}) \) that parametrizes the conic segment. In the 3D case the situation is different, since not every triangular patch on a quadric \( Q \) admits the canonical construction.

**Proposition 1.** Given a triangular portion \( \hat{Q} \) of a quadric \( Q \), the following statements are equivalent:

(i) \( \hat{Q} \) can be parameterized via a canonical construction of centre \( O \).

(ii) A point \( O \) exists, such that all nondegenerate boundaries of \( \hat{Q} \) pass through \( O \).

**Proof.** First consider the special case where all the boundaries are straight lines and, consequently, \( Q \) is planar. In this trivial case (ii) must be interpreted as “no restriction in the position of \( O \)” Indeed, we can always parameterize a triangular planar patch via a canonical construction of arbitrary centre \( O \). Simply take as starting triangle \( \Delta_c = \hat{Q} \) and do not carry out step (2) (the WRD).

In a general case, by construction, (i) \( \Rightarrow \) (ii). If (ii) is satisfied, consider the three rays from \( O \) to the corner points of \( \hat{Q} \), take a plane parallel to the plane \( T_0 \) tangent to \( Q \) at \( O \) and compute the three intersections between the rays and \( T_0 \). These three points define the starting triangle \( \Delta_c \) of a canonical construction. We can reproduce any given boundary via a WRD on the plane where it lies, starting from the corresponding side of \( \Delta_c \), which affects solely the Bézier points of this particular conic. The two planar WRD actions on a vertex of \( \Delta_c \) produce a common point for two boundaries, thereby coinciding on that vertex. Hence, the three planar WRD assemble into a three dimensional WRD. Thus we obtain a patch \( b(\mathbf{u}) \) on a quadric \( Q' \) with the same boundaries as the original patch. Clearly \( Q' = Q \), as both quadrics share at least 4 points (\( O \) and the three corner points of the patch) and the tangent planes at them. \( \square \)

**Remark.** Lodha and Warren (1990) developed a projective functional representation for quadrics, and derived a similar theorem where (i) reads that \( \hat{Q} \) can be represented in projective functional form. Obviously, this fact implies the equivalence of both methods, yet the WRD is again conceptually simpler.
5.4. Alternative representation of a linearly parameterized triangle

In step (1) of the canonical construction, we obtained an integral quadratic Bézier representation of a linearly parameterized triangle via standard degree-elevation. As an alternative, we may utilize a generalized degree elevation. Given a linearly parameterized triangle $\Delta_d$ of vertices $d_A, d_B, d_C$, we multiply and divide its Bézier form by an arbitrary linear function $\tilde{w}(u)$ and then divide by the unity:

$$
\begin{align*}
    d(u) &= \left( ud_A + vd_B + wd_C \right) \tilde{w}(u) / [u + v + w] \tilde{w}(u), \\
    \tilde{w}(u) &= \tilde{w}_A u + \tilde{w}_B v + \tilde{w}_C w.
\end{align*}
$$

(22)

Thus we obtain a rational quadratic Bézier patch $d(u)$ that parametrizes $\Delta_d$ with linear precision. We have inserted the base line $\tilde{w}(u) = 0$, which can be interpreted as a line on the plane $P$ defined by $\Delta_d$. This extended degree elevation has two degrees of freedom, corresponding to the location of the base line in $P$.

Note that the numerator of $d(u)$ (22) is a certain polynomial surface $p(u)$:

$$
p(u) = \left( ud_A + vd_B + wd_C \right) \tilde{w}(u).
$$

(23)

Rewriting it in the Bernstein basis:

$$
p(u) = up_A + vp_B + wp_C + 2uvp_{AB} + 2wp_{BC} + 2wp_{CA},
$$

(24)

and equating (23),(24) we get the corner points of $p(u)$:

$$
p_A = \tilde{w}_A d_A, \quad p_B = \tilde{w}_B d_B, \quad p_C = \tilde{w}_C d_C,
$$

and its inner points:

$$
2p_{AB} = \frac{\tilde{w}_B}{\tilde{w}_A} p_A + \frac{\tilde{w}_A}{\tilde{w}_B} p_B, \quad 2p_{BC} = \frac{\tilde{w}_C}{\tilde{w}_B} p_B + \frac{\tilde{w}_B}{\tilde{w}_C} p_C, \quad 2p_{CA} = \frac{\tilde{w}_A}{\tilde{w}_C} p_C + \frac{\tilde{w}_C}{\tilde{w}_A} p_A.
$$

5.5. Extended construction of patches on quadrics

As for curves, the extended degree-raising leads to a more general method (extended construction) for obtaining Bézier patches on quadrics. Let us apply an arbitrary WRD, of arbitrary centre $O$ not on the plane $P$ defined by $\Delta_c$, to the patch $d(u)$. In a coordinate system with origin $O$ this WRD does not modify the numerator of $d(u)$, and the resulting quadratic patch $b(u)$ can be written as:

$$
b(u) = \frac{p(u)}{u(u)}, \quad p(u) = \left( ud_A + vd_B + wd_C \right) \hat{w}(u),
$$

where $u(u)$ denotes a quadratic weight function, and $p(u)$ is the polynomial surface (23).

Clearly, at each border we reproduce the 2D extended construction of conic segments presented in Section 3.5. Therefore, the following conditions are satisfied (see Fig. 9):

1. The nondegenerate boundaries meet at a common point, the centre $O$; the straight line boundaries contain base points.

2. The lines $L_O$ of the three boundaries are coplanar (tangent line at $O$ for a nondegenerate boundary, line joining $O$ and the image $I$ of the base point for a straight line): these lines meet the plane $P$ at points on the base line, and hence lie on the plane $T_O$ defined by the base line and $O$. 


Next we give a simple demonstration confirming that these two conditions are sufficient for the Bézier patch $b(u)$ to lie on a quadric, and, in consequence, the extended construction indeed yields a patch on a quadric. Consider the three rays from $O$ to the corner points of...
b(u). Take an arbitrary plane parallel to \( T_0 \) and compute the three intersections with the rays. These three points define the starting triangle \( \Delta_c \) of a canonical construction. We can reproduce each boundary through the 2D canonical construction and thus obtain a patch on a quadric \( Q \) with the same control points \( b_i \) and boundaries than \( b(u) \). We could always reparameterize both patches so that they had unitary weights at the corner points (Albrecht, 1995) and hence share the same weights and control points. We conclude that \( b(u) \) coincides with the patch generated via the standard construction, up to a Möbius reparameterization, thereby lying on \( Q \). This reparameterization corresponds to the central projection that maps the triangle \( \Delta_c \) into \( \Delta_d \) (see Section 5.1).

The WRD \( b(u) \rightarrow d(u) \) is the inverse of a stereographic projection from \( Q \) onto the projection plane \( P \). It corresponds to a more general case, where \( P \) is not necessarily parallel to the tangent plane \( T_0 \) at the centre of projection \( O \). This stereographic projection provides a one-to-one map between \( Q \) and \( P \), except for the singular point \( O \), mapped to the singular or fundamental line, our base line in the domain space. As already noted by Teller and Séquin (1991), this line is the intersection of the planes \( P \) and \( T_0 \). In the canonical construction both planes are parallel and, consequently, the fundamental line lies at infinity. The extended degree elevation employed in the extended construction gives us two extra degrees of freedom, allowing us to set explicitly the location of the fundamental line.

6. Conditions for a quadratic patch to lie on a quadric

6.1. Conditions in terms of a WRD

Thanks to our investigation on the WRD construction, we are now ready to formalize in a precise manner the conditions for a quadratic Bézier patch to lie on a quadric.

**Theorem 2.** Given a quadratic Bézier patch \( b(u) \), the following statements are equivalent:

(i) \( b(u) \) lies on a quadric and \( u \rightarrow b(u) \) defines the inverse of a stereographic map.

(ii) \( b(u) \) admits an extended WRD construction with a certain centre \( O \).

(iii) (a) The nondegenerate boundaries of \( b(u) \) meet at a common point \( O \); those degenerate have a base point.

(b) The lines \( L_O \) of the three boundaries (tangent line at \( O \) for a nondegenerate boundary, line from \( O \) to the image of the base point for a straight line) are coplanar.

**Proof.** The implications (ii) \( \Rightarrow \) (iii) and (iii) \( \Rightarrow \) (i) were demonstrated in Section 5.5. It just remains to show that (i) \( \Rightarrow \) (ii). This is straightforward, as a projection is a particular case of a WRD, and in consequence the inverse map is also a WRD. \( \Box \)

**Remarks.** According to Niebuhr (1992), only in the case of degenerate quadrics \( Q \) (those with Gaussian curvature \( K = 0 \), i.e., cones and cylinders), there exist Bézier patches on \( Q \).
that do not define a stereographic projection. From this fact and the equivalence of (i), (ii) in Theorem 2, it follows that:

- The WRD construction generates all possible quadratic patches on nondegenerate quadrics.
- Theorem 2 gives conditions sufficient for a patch \( b(u) \) to lie on a quadric, and necessary to lie on a nondegenerate quadric.

The third condition listed in the Introduction, that the boundaries assume the parameter value \( \pm \infty \) at \( O \), is fulfilled only by the canonical construction. We conclude that such a condition is not necessary. In fact, it can be always met after a certain reparametrization, if (iii) is actually satisfied. Simply consider that in this case we can generate the patch via either the extended construction, according to (ii), or the canonical construction (Proposition 1). These two patches are related by a Möbius reparametrization, as justified in Section 5.5.

### 6.2. A simple procedure to identify patches on quadrics

In this Section we show how to translate condition (iii) of Theorem 2 into two practical, easy to check relationships.

First we are to compute the candidate point \( O \). In case the three boundaries of \( b(u) \) are straight lines, \( b(u) \) is planar. Therefore, if (ii) is satisfied for a certain \( O \) then \( b(u) \) has minimum degree one and, consequently, admits a WRD construction with arbitrary point \( O \). Thus, we can choose any point \( O \) (not on the plane defined by \( b(u) \)).

In a nonplanar case, the only candidate point \( O \) is the intersection of the boundary planes \( (u = 0, v = 0, w = 0) \), defined by triples of boundary control points. However, one boundary conic may degenerate into a straight line, even two boundaries in the case of a doubly ruled quadric (one sheeted hyperboloid or hyperbolic paraboloid). Then the corresponding boundary plane is ill defined. Nevertheless, if \( b(u) \) indeed lies on a quadric, then any line in parameter space intersects the fundamental line, and hence its image passes through \( O \). Therefore one may use, as a replacement for a degenerate boundary, such a line in the parameter space, for instance a radial line joining a vertex with the midpoint of the opposite edge. No more than two straight lines can pass through a regular point of a nonplanar quadric. Hence, no more than two radial lines through a vertex, which includes the boundaries, can degenerate into straight lines. In fact the candidate point \( O \) can be calculated as the intersection of any three planes determined by boundaries or radial lines, planes defined by triples of control points. This wide choice of planes helps us compute \( O \) in a numerically reliable way.

Once \( O \) has been computed, we must determine if the condition (iii(a)) is satisfied. Just express \( b(u) \) in a coordinate system with origin at \( O \), compute the homogeneous points \( p_i = w_i b_i \) and check condition (16) for each boundary conic. In the case of \( O \) at infinity, for instance the \( z \)-direction, we calculate points \( p_i = w_i z_i \), where \( z_i \) is the projection of \( b_i \) onto the plane \( z = 1 \) perpendicular to the direction defined by \( O \). If \([\alpha \beta \gamma] \) denote coordinates in the basis \([p_A, p_B, p_C] \), then those of the interior points \( p_{AB}, p_{BC}, p_{CA} \) satisfy the relationships:

\[
\alpha_{AB} \beta_{AB} = \beta_{BC} \gamma_{BC} = \gamma_{CA} \alpha_{CA} = 1.
\]  

(25)
If condition (iii(a)) is fulfilled, according to expression (18) the lines $L_O$ through $O$ in condition (iii(b)) have vectors of coordinates:

\[
\begin{align*}
w = 0: & \quad [ -\alpha_{AB} \ \beta_{AB} \ 0 ], \\
u = 0: & \quad [ 0 \ -\beta_{BC} \ \gamma_{BC} ], \\
v = 0: & \quad [ \alpha_{CA} \ 0 \ -\gamma_{CA} ].
\end{align*}
\]

These vectors are coplanar if and only if their determinant vanishes:

\[
\alpha_{AB}\beta_{BC}\gamma_{CA} = \beta_{AB}\gamma_{BC}\alpha_{CA}.
\]

Observe the simplicity and cyclic symmetry of the equalities (25), (26) to check, which are tantamount to (iii) in Theorem 2.

7. Quadric patches on cylinders and cones

In this section we investigate how to utilize a WRD for constructing the aforementioned patches on degenerate quadrics (cones and cylinders) that do not define a stereographic projection. In essence we reinterpret in terms of a WRD, describing a constructive method, a result given in (Boehm and Hansford, 1991; Boehm, 1993).

A cone $Q$ is defined by a conic $\Gamma$ (directrix) and a point $O$ (vertex). Assume we have a segment of $\Gamma$ represented in Bézier form as $b(u)$, corresponding to the boundary $v = 0$ of our triangular patch on $Q$. We apply the following WRD-based method, illustrated in Fig. 10:

1. Duplicate the Bézier curve and subdivide the overlapping copy at arbitrary parameter value $u_0 \in [0, 1]$. This operation yields two curves, the boundaries $u = 0$, $w = 0$, of a degenerate triangular patch.

2. Stretch this patch by applying a WRD with centre $O$. The resulting surface lies on $Q$, as in a WRD all points move along rays through $O$.

![Fig. 10. Constructing a triangular patch on a cone via WRD.](image-url)
As subdividing a Bézier curve does not modify the location of the point corresponding to \( \infty \), the three boundaries have such points on a common ruling. However, this property no longer holds if we reparametrize the patch after step (1) or, equivalently, if we reparameterize the boundaries after step (1).

Finally, observe that, in the particular case where \( O \) lies at infinity, this WRD construction produces a cylinder.

8. Conclusions

The WRD (Weighted Radial Displacement) is a powerful constructive tool that provides an accessible and geometric look at Bézier conics and quadrics. If we take a straight segment or a triangle and modify it via WRD of arbitrary centre \( O \), we define a perspective central projection. If we degree elevate the segment or triangle and then apply a WRD, we obtain a conic or quadric through \( O \). The inverse of such a WRD corresponds to a stereographic projection of centre \( O \). Any conic segment in quadratic Bézier form, or any quadratic Bézier patch on a nondegenerate quadric admits this WRD-based construction, which in addition gives a clear geometric interpretation of the Möbius reparametrization. The WRD also leads to a method for defining Bézier patches on degenerate quadrics (i.e., cones and cylinders), thereby providing a novel, unified treatment of quadrics.

A quadratic patch \( b(u) \) lies on a quadric if the following conditions are met:
(a) The nondegenerate boundaries of \( b(u) \) meet at a common point \( O \); those degenerate have a base point.
(b) The lines \( L_o \) of the three boundaries are coplanar. Such lines are the tangents at \( O \) for nondegenerate boundaries, and lines from \( O \) to the image of the base point for straight lines.

These two conditions translate into very simple numerical relationships to be satisfied by the homogeneous control points of the patch. Another condition appearing in some references, that the boundaries take the parameter values \( \pm \infty \) at \( O \), is unnecessary. It is satisfied for a particular case of the WRD construction, and can be always met if we reparameterize the patch.

Conics and quadrics are the particular quadratic instance of a wider class of curves and surfaces, called monoids. Most of the ideas exposed in this paper carry over to curves and surfaces of higher degree. If we degree-elevate a curve of arbitrary degree, by inserting a base point \( u_0 \), and then apply a WRD of centre \( O \), we get a new curve that passes through \( O \) for \( u = u_0 \). Clearly, inserting several base points, followed by a WRD, furnishes a monoidal curve, and a similar procedure holds for constructing monoidal surfaces. Research is under way on such monoids.

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