On surface normal and Gaussian curvature approximations
given data sampled from a smooth surface

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Abstract

Approximations to the surface normal and to the Gaussian curvature of a smooth surface are often required when the surface is defined by a set of discrete points. The accuracy of an approximation can be measured using asymptotic analysis. The errors of several approximations to the surface normal and to the Gaussian curvature are compared.

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1. Introduction

Approximations to the surface normal and to the Gaussian curvature of a smooth surface are often required when the surface is defined by a set of discrete points rather than by a formula. This situation may arise in reverse engineering when accurate measurements are made of the surface of a solid object. A standard technique in numerical analysis for comparing approximations is asymptotic analysis. It is used here to compare the errors of several approximations to the surface normal and to the Gaussian curvature. Below, the term curvature is used to mean Gaussian curvature.

Three classes of methods that help one approximate information about a surface are as follows: one may discretize a mathematical formula that gives the information for continuous surfaces; one may interpolate the surface with a simpler surface for which the information can be calculated; and one may approximate the surface in a least squares

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sense by a simpler surface for which the information can be calculated. This paper is concerned with discretization and interpolation methods. These methods are appropriate when data from the surface is highly accurate and the surface is smooth. The accuracy of these methods can be studied with asymptotic error analysis. Least squares methods are appropriate for noisy data but asymptotic error analysis does not apply to them. For examples of least squares methods see Abdelmalek (1990), Flynn (1989), Kent (1994), Stokey (1992), Todd (1986), and Yokoya (1989).

The new features of this work include a new method for approximating the curvature by discretizing the spherical image method (Li, 1997), and a comparison of well-known methods through asymptotic error analysis. The asymptotic error of normal and curvature approximations appears not to have been published previously. One surprising result is that the angle deficit method, which is a very popular method in the literature, is not necessarily very accurate.

Two arrangements of discrete points on a smooth surface are considered. The points will be called non-uniform data if they lie on a smooth surface \( z = f(x, y) \) but the \((x, y)\) points have no particular pattern. The points will be called uniform data if they lie on a smooth surface \( z = f(x, y) \) and the \((x, y)\) points form a square grid.

For the analysis and numerical examples, the approximations are found using data taken near a given point in a way that depends on a parameter \( h \). The error is expressed as a power series in ascending powers of \( h \). The leading term of the power series indicates the behaviour of the error as the points become close. Two formulas can be compared by examining the leading powers of \( h \) in their asymptotic errors. Three disadvantages of the asymptotic approach are: it may not be informative in practice if one is presented with a set of points and cannot generate a finer mesh; when the data points are not close, it is possible that an asymptotically low accuracy formula is more accurate than an asymptotically high accuracy formula; and it is not possible to study least squares methods with asymptotic analysis.

2. Preliminaries

2.1. Formula for a surface

A general form for a surface is \( f(x, y, z) = 0 \). Assume the surface is smooth and has a normal existing everywhere. At any the point on the surface, imagine a coordinate system with its \( z \)-axis roughly parallel to the normal to the surface at that point. The surface near that point can be expressed in the form \( z = f(x, y) \). This local form can be achieved numerically for a given set of points by estimating the normal and projecting the points onto a plane with that normal. Below, it is assumed that the points are from a smooth surface of the form \( z = f(x, y) \). Without loss of generality, assume that the point where the normal and curvature are to be approximated is the origin. A given surface can always be translated so this is the case. The function \( f(x, y) \) can be expanded by Maclaurin series as

\[
\begin{align*}
  f(x, y) &= f_x(0, 0)x + f_y(0, 0)y + \frac{f_{xx}(0, 0)}{2} x^2 + f_{xy}(0, 0)xy \\
  &\quad + \frac{f_{yy}(0, 0)}{2} y^2 + \cdots.
\end{align*}
\]

(2.1)
2.2. Normal and curvature of the surface at the origin

Define the surface by the vector

\[ S(x, y) = \begin{pmatrix} x \\ y \\ f(x, y) \end{pmatrix}, \]  

(2.2)

where \( f(x, y) \) is given in (2.1), and use the procedure described in (Farin, 1997, p. 348) to calculate the normal and curvature. The unit normal to (2.2) at the origin is

\[ N = \frac{1}{\sqrt{L}} \begin{pmatrix} -f_x(0, 0) \\ -f_y(0, 0) \\ 1 \end{pmatrix}, \quad L = f_x(0, 0)^2 + f_y(0, 0)^2 + 1. \]  

(2.3)

The curvature of (2.2) at the origin is

\[ k = \frac{f_{xx}(0, 0)f_{yy}(0, 0) - f_{xy}(0, 0)^2}{L^2}. \]  

(2.4)

2.3. Asymptotic error analysis

Asymptotic errors will be expressed with the \( O \) notation; \( O(h^p) \) stands for a power series in ascending powers of \( h \) that starts with \( h^p \) or a power of \( h \) higher than \( p \).

Calculations in the error analysis are simplified by noting that any surface of the form (2.2) can be rotated so that the normal at the origin is parallel to the \( z \)-axis. This rotation will cause the linear terms in \( x \) and \( y \) to disappear. Thus, for the error analysis (but not for numerical examples) it will be assumed that the surface is of the form

\[ g(x, y) = \frac{B_{20}}{2} x^2 + B_{11} xy + \frac{B_{02}}{2} y^2 + \frac{B_{30}}{6} x^3 + \frac{B_{21}}{2} x^2 y + \frac{B_{12}}{2} xy^2 + \cdots \]  

or

\[ R(x, y) = \begin{pmatrix} x \\ y \\ g(x, y) \end{pmatrix}. \]  

(2.5)

(2.6)

The normal and curvature at the origin of \( R(x, y) \) are

\( (0, 0, 1)^T \) and \( B_{20}B_{02} - B_{11}^2 \).  

(2.7)

2.4. Points on the \( R(x, y) \) near the origin

Let \( P_i, i = 1, 2, \ldots, n \), be \( n \) points on \( R(x, y) \) near the origin \( O \); let \( n_i \) and \( n_0 \) be approximations to the unit normals to the surface at \( P_i \) and at \( O \) (see Fig. 1). The positive indices are considered cyclically so index \( n + 1 \) is the same as index 1 and index \( i \) is the same as index \( (i - 1) \mod n + 1 \). If \( P_i \) has \( x \) and \( y \) coordinates \( x_i = a_ih \) and \( y_i = b_ih \), then

\[ P_i = \begin{pmatrix} a_ih \\ b_ih \\ z_ih^2 + O(h^3) \end{pmatrix}, \quad i = 1, 2, \ldots, n, \]  

(2.8)
where

\[ z_i = \frac{B_{20}}{2}a_i^2 + B_{11}a_i b_i + \frac{B_{02}}{2}b_i^2. \]  

(2.9)

The points can also be expressed in polar coordinates with \( x \) and \( y \) coordinates \( x_i = r_i (\cos \theta_i)h \) and \( y_i = r_i (\sin \theta_i)h \) so

\[ P_i = r_i \begin{pmatrix} \cos \theta_i h \\ \sin \theta_i h \end{pmatrix} + O(h^3). \]  

(2.10)

where

\[ w_i = \frac{B_{20}}{2} \cos^2 \theta_i + B_{11} \cos \theta_i \sin \theta_i + \frac{B_{02}}{2} \sin^2 \theta_i. \]  

(2.11)

Assume that the points \( P_i \) are taken in a generally counterclockwise order so that \( 0 \leq \theta_1 \leq \theta_2 \leq \cdots \leq \theta_n < 2\pi \). Let the normal to the triangle \( P_i O P_{i+1} \) be

\[ n_{i,i+1} = (P_i - O) \times (P_{i+1} - O) = P_i \times P_{i+1}, \]

\[ = r_i r_{i+1}^2 h^2 \begin{pmatrix} (w_i + w_{i+1} \sin \theta_i - w_i \sin \theta_{i+1})h + O(h^2) \\ (-w_{i+1} \cos \theta_i + w_i \cos \theta_{i+1})h + O(h^2) \end{pmatrix}. \]  

(2.12)

Some other useful formulas are:

\[ \|P_i\| = r_i h \left( 1 + \frac{w_i^2}{2} h^2 + O(h^3) \right). \]

\[ \|P_i \times P_{i+1}\| = \|n_{i,i+1}\| = r_i r_{i+1} \sin(\theta_{i+1} - \theta_i) h^2 \]  

\[ \times \left( 1 + \frac{w_i^2 + w_{i+1}^2}{2} \frac{\cos(\theta_{i+1} - \theta_i)}{\sin^2(\theta_{i+1} - \theta_i)} + O(h^3) \right). \]
The area of triangle $P_i O P_{i+1}$ is
\[
S_{i,i+1} = \frac{1}{2} \left\| (P_i - O) \times (P_{i+1} - O) \right\| = \frac{1}{2} \| P_i \times P_{i+1} \|
\]
\[
= \frac{r_ir_{i+1}}{2} \sin(\theta_{i+1} - \theta_i)h^2 + O(h^4).
\]

Let $\theta_{i,i+1}$ be the planar angle $P_i O P_{i+1}$ and let it be positive by convention. $\theta_{i,i+1}$ satisfies
\[
\sin \theta_{i,i+1} = \frac{\| P_i \times P_{i+1} \|}{\| P_i \| \| P_{i+1} \|} = \sin(\theta_{i+1} - \theta_i)
\]
\[
\times \left( 1 - \frac{2w_iw_{i+1} - (w_i^2 + w_{i+1}^2)\cos(\theta_{i+1} - \theta_i)}{2\sin^2(\theta_{i+1} - \theta_i)}\cos(\theta_{i+1} - \theta_i)h^2 + O(h^3) \right).
\]

(2.14)

3. Methods

3.1. Quadratic fit method for normal and curvature approximation

A surface can be approximated by an interpolating quadratic surface. The normal and curvature of the quadratic surface are approximations to the normal and curvature of the surface.

A quadratic surface that passes through the origin is
\[
z = A_{10}x + A_{01}y + \frac{A_{20}}{2}x^2 + A_{11}xy + \frac{A_{02}}{2}y^2.
\]

(3.1)

The requirement that it pass through five other given points $(X_i, Y_i, Z_i), i = 1, 2, \ldots, 5,$ is expressed by the linear system
\[
\begin{bmatrix}
X_1 & Y_1 & \frac{x_1^2}{2} & X_1Y_1 & \frac{y_1^2}{2} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
X_5 & Y_5 & \frac{x_5^2}{2} & X_5Y_5 & \frac{y_5^2}{2}
\end{bmatrix}
\begin{bmatrix}
A_{10} \\
A_{01} \\
A_{20} \\
A_{11} \\
A_{02}
\end{bmatrix}
= \begin{bmatrix}
Z_1 \\
Z_2 \\
Z_3 \\
Z_4 \\
Z_5
\end{bmatrix}.
\]

(3.2)

If this system has a unique solution, the unit normal and curvature of the quadratic (3.1) at the origin are
\[
\frac{1}{\sqrt{L}} \begin{bmatrix}
-A_{10} \\
-A_{01} \\
1
\end{bmatrix} \quad \text{and} \quad \frac{A_{20}A_{02} - A_{11}^2}{L^2}, \quad L = A_{10}^2 + A_{01}^2 + 1.
\]

(3.3)

A variation of this method is discussed in Todd (1986). There can be difficulties in fitting the quadratic (3.1) (Lancaster, 1986, p. 135) and occasionally it gives poor accuracy. The poor accuracy may be due to the matrix on the left of (3.2) being badly conditioned. The form of quadratic may be extended to include terms in $x^2y, xy^2,$ and $x^2y^2,$ in which case eight points would be required and an $8 \times 8$ system of linear equations would result.
3.2. Average of normals for normal approximation

A surface can be approximated by a set of triangular faces. The unit normal to the surface at a given point can be approximated by the unit vector parallel to an average of unit normals of the triangular faces that surround that point. Various averages have been used: the arithmetic mean, an area-weighted average, and an angle-weighted average (Akima, 1984; Brown, 1991; Gouraud, 1971; Shirman, 1987).

3.3. Spherical image method for curvature approximation

The curvature of a surface can be approximated by the spherical image method. This method is an obvious discretization of a theorem for defining curvature due to Rodrigues (Kreysig, 1991, p. 187; Hilbert, 1952). Imagine traversing a closed path on a surface around a given point on the surface. If the tails of the unit normals to the surface along the path are placed at the origin, the heads of those unit normals trace out a closed curve on a unit sphere called the spherical image of the path on the surface. The theorem states that the curvature at the given point on a surface is the limit of the area of the spherical image of the path divided by the area of the path as the path shrinks around the point. For example, a path on
\( \mathbf{R}(x, y) \) around \( \mathbf{O} \) is approximated by the non-planar polygon \( P_1 P_2 \ldots P_n P_1 \) (see Fig. 2 for an example with \( n = 5 \)). The spherical image of this path will be approximated by the spherical polygon \( n_1 n_2 \ldots n_i n_1 \) on the unit sphere (\( n_i \) is an approximation to the unit normal at \( P_i \)). The ratio of the area of all the triangles \( n_i n_0 n_{i+1} \) over the area of all the triangles \( P_i O P_{i+1} \) is an approximation to the curvature at \( \mathbf{O} \).

3.4. Angle deficit method for curvature approximation

The curvature of a surface can be approximated by another method developed from the theorem of Rodrigues (Kreysig, 1991, p. 187; Hilbert, 1952). The surface can be approximated by a polyhedron with triangular faces whose vertices are points on the surface. For example, the point \( \mathbf{O} \) is surrounded by the triangular faces \( P_i O P_{i+1} \) (see Fig. 3). The spherical image of the polyhedron is a set of points on the unit sphere (the heads of unit vectors parallel to \( n_i, i+1 \)). One can join these points by arcs of great circles to

Fig. 3. Angle deficit method.
form a spherical polygon on the unit sphere. The area of the spherical polygon is the angle deficit of the polyhedron, $2\pi - \sum \theta_{i,i+1}$; this surprising result is given in (Calladine, 1986). The area of each triangular face of the polyhedron can be partitioned into three equal parts, one corresponding to each of its vertices, so that the total area related to point $O$ on the polyhedron is $\frac{1}{4} \sum S_{i,i+1}$. This value is taken as an approximation to the area of a path on the surface around $O$, although the path has not been specified. An approximation to the curvature at $O$ is then

$$k = \frac{2\pi - \sum \theta_{i,i+1}}{\frac{1}{4} \sum S_{i,i+1}}. \tag{3.4}$$

Formula (3.4) is often quoted as a formula for approximating curvature (Azariadis, 1997; Calladine, 1983, p. 144; Daniel, 1997; Hinds, 1991; Lin, 1982; Regge, 1961; Sorkin, 1975; Stokey, 1992).

4. Results for surfaces described by non-uniform data

**Lemma 4.1.** For non-uniform data, the quadratic fit method approximates the unit normal to accuracy $O(h^2)$ and the curvature to accuracy $O(h)$.

**Proof.** Making the quadratic (3.1) interpolate the five points $P_i, i = 1, 2, \ldots, 5$, in (2.8) on $R(x, y)$ means each equation of (3.2) is of the form

$$a_i A_{10} + b_i A_{01} + \frac{a_i^2 h}{2} A_{20} + a_i b_i h A_{11} + \frac{b_i^2 h}{2} A_{02} = z_i h + O(h^2).$$

Let the determinant $D$ be

$$D = \det(a_i, b_i, a_i^2, a_i b_i, b_i^2) \tag{4.1}$$

(the determinant of the matrix in (3.2) is $h^8 D$). The columns of $D$ are independent for general $a_i$ and $b_i$; hence, generally $D$ is non-singular. However, $z_i$ is a linear combination of $a_i^2$, $a_i b_i$, and $b_i^2$ from (2.9) so a determinant that includes the column $z_i$ can be simplified. Cramer’s rule gives $A_{10}$ as

$$A_{10} = \frac{\det(z_i h + O(h^2), b_i, a_i^2, a_i b_i, b_i^2)}{D} = \frac{\det(O(h^2), b_i, a_i^2, a_i b_i, b_i^2)}{D} = O(h^2).$$

Similarly, $A_{01} = O(h^2)$.

Cramer’s rule gives $A_{20}$ as

$$A_{20} = \frac{\det(a_i, b_i, z_i + O(h), a_i b_i, b_i^2)}{D},$$

and using (2.9),
\[ A_{20} = \frac{\det(a_i, b_i, B_{20} a_i^2, a_i b_i, \frac{h^2}{2}) + \det(a_i, b_i, O(h), a_i b_i, \frac{h^2}{2})}{D} = B_{20} D + O(h) = B_{20} + O(h). \]

Similarly, \( A_{11} = B_{11} + O(h) \) and \( A_{02} = B_{02} + O(h) \).

From (3.3),
\[ L = 1 + O(h^4), \]

thus the unit normal of the quadratic (3.1) is from (3.3)
\[ N = \left( \begin{array}{c} O(h^2) \\ O(h^2) \\ 1 + O(h^4) \end{array} \right), \]

which is an \( O(h^2) \) approximation to the exact unit normal given in (2.7). The curvature of the quadratic (3.1) is from (3.3) \( B_{20} B_{02} - B_{11}^2 + O(h) \), which is an \( O(h) \) approximation to the exact curvature given in (2.7). □

For Lemmas 4.2, 4.3, and 4.4, assume that at least three distinct points \( P_i, i = 1, 2, \ldots, n, n \geq 3 \), on \( R(x, y) \) are chosen such that each \( \theta_{i,i+1} \) of (2.14) is smaller than \( \pi \).

**Lemma 4.2.** For non-uniform data, the unit vector parallel to the arithmetic mean of unit normals of the triangular faces around a point approximates the unit normal to the surface at that point to accuracy \( O(h) \).

**Proof.** Using points \( P_i \) in (2.10) on \( R(x, y) \), the normals to the triangular faces around the origin are from (2.12)
\[ n_{i,i+1} = r_i r_{i+1} h^2 \begin{pmatrix} O(h) \\ O(h) \\ \sin(\theta_{i,i+1} - \theta_i) \end{pmatrix}. \]

The unit normals of the triangular faces are \( (O(h), O(h), 1 + O(h^2))^T \). Any linear combination of these unit normals, and specifically the arithmetic mean, will give an \( O(h) \) approximation to the exact normal \( (0, 0, 1)^T \) in (2.3). □

Numerical tests indicate that the arithmetic mean, an area-weighted average, and an angle-weighted average are all no higher accuracy than \( O(h) \). An open question is to find a linear combination of the normals of the triangular faces, based on geometric considerations, that approximates the normal of the surface to \( O(h^2) \) accuracy.

Note that the normals calculated in Lemma 4.2 are not accurate enough to be used in the next lemma. \( O(h^2) \) accurate normals can be obtained by extrapolation with normals calculated with a certain \( h \) and with \( 2h \) as long as the points involved are scaled versions of each other. If \( N(h) \) and \( N(2h) \) are \( O(h) \) approximations to the normal, the unit vector parallel to \( 2N(h) - N(2h) \) is an \( O(h^2) \) approximation to the unit normal (Faires and Burden, 1993, p. 112). This approach was used in the numerical example shown in Table 1.
Lemma 4.3. For non-uniform data and with unit normals known to accuracy $O(h^2)$, the spherical image method approximates the curvature to accuracy $O(h)$. 

Proof. Since the surface normal at $P_i$ is parallel to $R_x(x_i, y_i) \times R_y(x_i, y_i)$, or $(-f_x(x_i, y_i), -f_y(x_i, y_i), 1)^T$, the exact unit normal to $R(x, y)$ at $P_i$ is

$$N_i = \left( \frac{-(B_{20} \cos \theta_i + B_{11} \sin \theta_i) r_i h + O(h^2)}{1 + O(h^2)} \right). \quad (4.2)$$

Let the approximate unit normal at $P_i$ be

$$n_i = \left( \frac{-(B_{20} \cos \theta_i + B_{11} \sin \theta_i) r_i h + O(h^2)}{1 + O(h^2)} \right),$$

where the $n_i$ are $O(h^2)$ approximations to the exact values $N_i$ in (4.2) so the $O(h^2)$ terms in $n_i$ are not necessarily the same as the $O(h^2)$ terms in $N_i$. Let the approximation to the unit normal at the origin be

$$n_0 = \left( \begin{array}{c} O(h^2) \\ O(h^2) \\ 1 + O(h^2) \end{array} \right).$$

The signed area of the spherical image triangle on the unit sphere is approximately the area of the corresponding planar triangle formed from points on the unit sphere (the exact area of the spherical triangle can be used, but in numerical tests, resulting curvature seems to be the same order of accuracy as with the planar approximation)

$$I_{i,i+1} = \pm \frac{1}{2} \| (n_i - n_0) \times (n_{i+1} - n_0) \| = [B_{20} B_{02} - B_{11}^2] \frac{r_{i+1} r_i}{2} \sin(\theta_{i+1} - \theta_i) h^2 + O(h^3),$$

where the sign is in the $B_{20} B_{02} - B_{11}^2$ factor. Using the above equation and (2.13), the spherical image approximation to the curvature is

$$\sum \frac{I_{i,i+1}}{S_{i,i+1}} = B_{20} B_{02} - B_{11}^2 + O(h),$$

which is an $O(h)$ approximation to the exact value given in (2.7). 

Lemma 4.4. For non-uniform data, the angle deficit method approximates the curvature to accuracy $O(1)$. 

Proof. From (2.14), $\sin \theta_{i,i+1}$ is of the form

$$\sin \theta_{i,i+1} = \sin(\theta_{i+1} - \theta_i)(1 - Ah^2 + O(h^3)) \quad (4.3)$$

where $A$ can be identified in (2.14). Suppose, $\theta_{i,i+1} = B + C h + D h^2 + O(h^3)$, then

$$\sin \theta_{i,i+1} = \sin B + (C \cos B) h + \left( D \cos B - \frac{C^2}{2} \sin B \right) h^2 + O(h^3).$$
Matching coefficients in (4.3), \( B = \theta_{i+1} - \theta_i, C = 0, \) and \( D = -\tan(\theta_{i+1} - \theta_i) A. \) Thus,
\[
\theta_{i,i+1} = \theta_{i+1} - \theta_i - \frac{2w_iw_{i+1} - (w_i^2 + w_{i+1}^2) \cos(\theta_{i+1} - \theta_i)}{2 \sin(\theta_{i+1} - \theta_i)} h^2 + O(h^3).
\]

Now the angle deficit is
\[
2\pi - \sum_{i=1}^{n} \theta_{i,i+1} = 2\pi - \sum_{i=1}^{n} (\theta_{i+1} - \theta_i)
\]
\[
+ \sum_{i=1}^{n} \frac{2w_iw_{i+1} - (w_i^2 + w_{i+1}^2) \cos(\theta_{i+1} - \theta_i)}{2 \sin(\theta_{i+1} - \theta_i)} h^2 + O(h^3)
\]
\[
= \sum_{i=1}^{n} \frac{2w_iw_{i+1} - (w_i^2 + w_{i+1}^2) \cos(\theta_{i+1} - \theta_i)}{2 \sin(\theta_{i+1} - \theta_i)} h^2 + O(h^3).
\]

(4.4)

One third of the total area of the triangles \( P_i OP_{i+1} \) is from (2.13),
\[
\frac{1}{3} \sum_{i=1}^{n} S_{i,i+1} = \frac{1}{6} \sum_{i=1}^{n} (r_i r_{i+1} \sin(\theta_{i+1} - \theta_i)) h^2 + O(h^4).
\]

(4.5)

Following (3.4), the approximation to the curvature is the quotient of (4.4) and (4.5). To show that the constant term in that quotient is not always equal to the exact curvature \( B_{20}B_{20}^{-1} \), take \( r_i = 1, \) let \( \theta = 2\pi / n, \theta_i = (i-1)\theta, \) \( B_{20} = B_{02} = 2, \) and \( B_{11} = 0. \) In this special case \( w_i = 1 \) (2.11) and the quotient (3.4) with (4.4) and (4.5) reduces to
\[
\frac{\sum_{i=1}^{n} 2 - 2\cos\theta - 2\sin\theta}{\frac{2}{3} \sum_{i=1}^{n} \sin\theta} + O(h) = \frac{6}{1 + \cos\theta} + O(h).
\]

The exact curvature is 4, so the angle deficit does not give the correct constant term for this example. \( \square \)

5. Results for surfaces described by uniform data

For this section on uniform data, let the eight neighbouring points of \( O \) be \( P_1, P_2, \ldots, P_8 \) as shown in Fig. 4.

Although a quadratic of the form (3.1) can be found for uniform data, it is probably more sensible to use finite difference formulas. Finite difference formulas can be derived from quadratic fits, although the results are not exactly what one would obtain from the form (3.1). The advantage of finite difference formulas is that they do not require the solution of a system of equations. A similar idea to the finite difference approach called the Cross Patch method is proposed in Stokey (1992).

**Lemma 5.1.** For uniform data, finite difference formulas result in an approximation to the unit normal to accuracy \( O(h^2) \) and an approximation to the curvature to accuracy \( O(h^2) \).
Proof. The partial derivatives of \( R(x, y) \) can be approximated with \( O(h^2) \) finite difference formulas such as

\[
\frac{P_1 - P_5}{2h} = R_x(0, 0) + \begin{pmatrix} O(h^2) \\ O(h^2) \end{pmatrix} = \begin{pmatrix} 1 + O(h^2) \\ 0 \end{pmatrix},
\]

\[
\frac{P_1 - 2O + P_5}{h^2} = R_{xx}(0, 0) + \begin{pmatrix} O(h^2) \\ O(h^2) \end{pmatrix} = \begin{pmatrix} O(h^2) \\ O(h^2) \end{pmatrix},
\]

and

\[
\frac{P_2 - P_8 - P_4 + P_6}{4h^2} = R_{xy}(0, 0) + \begin{pmatrix} O(h^2) \\ O(h^2) \end{pmatrix} = \begin{pmatrix} O(h^2) \\ 0 \end{pmatrix}.
\]

The approximation to the normal is obtained as in (2.3) from

\[
\frac{P_1 - P_5}{2h} \times \frac{P_1 - P_7}{2h}
\]

and the curvature as in (2.4). It is easy to see that the errors for both are \( O(h^2) \).

In calculation, it is not necessary to use the \( h \) in the finite difference formulas. One works with expressions such as \( P_1 - P_5 \) to calculate partial derivatives scaled by multiples of \( h \). The normalizing of a vector and the calculation of the curvature from (2.4) cause the factors of \( h \) to cancel.

The approximation to the normal with finite differences in Lemma 5.1 is the same order as a quadratic fit with non-uniform data, Lemma 4.1, and the approximation to the curvature with finite differences is one order higher than a quadratic fit with non-uniform data, Lemma 4.1.

Lemma 5.2. For uniform data, the unit vector parallel to an area-weighted average of unit normals of the triangular faces around a given point approximates the unit normal at that point to accuracy \( O(h^2) \).
Proof. Choose the four points on the $\mathbf{R}(x, y)$, $\mathbf{P}_1$, $\mathbf{P}_3$, $\mathbf{P}_5$, and $\mathbf{P}_7$ (see Fig. 4). Note the identity for vectors $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_3$, and $\mathbf{V}_4$.

$$\mathbf{V}_1 \times \mathbf{V}_2 + \mathbf{V}_2 \times \mathbf{V}_3 + \mathbf{V}_3 \times \mathbf{V}_4 + \mathbf{V}_4 \times \mathbf{V}_1 = (\mathbf{V}_1 - \mathbf{V}_3) \times (\mathbf{V}_2 - \mathbf{V}_4).$$

Using a notation similar to that in (2.12), the above vector identity gives

$$n_{13} + n_{35} + n_{57} + n_{71} = (\mathbf{P}_1 - \mathbf{P}_5) \times (\mathbf{P}_3 - \mathbf{P}_7) = O(h^2) \begin{pmatrix} O(h^2) \\ O(h^2) \\ 1 \end{pmatrix}.$$ 

The unit vector parallel to the above vector is an $O(h^2)$ approximation to the exact normal in (2.7). \(\square\)

Numerical tests indicate that other combinations of the unit normals such as the arithmetic mean and an angle-weighted average also approximate the unit normal to the surface to accuracy $O(h^2)$. The result in Lemma 5.2 could be used with the spherical image method as in Lemma 4.3 to give the curvature to accuracy $O(h)$. However, with uniform data, the curvature is approximated to accuracy $O(h^2)$ by the spherical image method as shown in Lemma 5.3 below.

Lemma 5.3. For uniform data, the spherical image method approximates the curvature to accuracy $O(h^2)$.

Proof. The area enclosed by the path on the surface $\mathbf{P}_1 \mathbf{P}_3 \mathbf{P}_5 \mathbf{P}_7 \mathbf{P}_1$ (see Fig. 4) is approximately from (2.13)

$$\sum_i S_{i,i+2} = 2h^2 + O(h^4).$$  \hspace{1cm} (5.1)

where the index $i$ is 1, 3, 5, and 7 taken in a cyclic manner. The normals can be approximated as in Lemma 5.2. For example, the unit normal at $\mathbf{P}_1$ is approximated by

$$n_1 = \frac{(\mathbf{R}(2h, 0) - \mathbf{O}) \times (\mathbf{R}(h, h) - \mathbf{R}(h, -h))}{\| (\mathbf{R}(2h, 0) - \mathbf{O}) \times (\mathbf{R}(h, h) - \mathbf{R}(h, -h)) \|}.$$  \hspace{1cm} (5.2)

The approximations to the unit normals at $\mathbf{O}$ (from Lemma 5.2), $\mathbf{P}_1$, $\mathbf{P}_3$, $\mathbf{P}_5$, and $\mathbf{P}_7$ are

$$n_0 = \begin{pmatrix} O(h^2) \\ O(h^2) \\ 1 + O(h^4) \end{pmatrix}, \quad n_1 = \begin{pmatrix} -B_{20}h + O(h^2) \\ -B_{11}h + O(h^2) \\ 1 + O(h^2) \end{pmatrix},$$

$$n_3 = \begin{pmatrix} -B_{11}h + O(h^2) \\ -B_{02}h + O(h^2) \\ 1 + O(h^2) \end{pmatrix}, \quad n_5 = \begin{pmatrix} B_{20}h + O(h^2) \\ B_{11}h + O(h^2) \\ 1 + O(h^2) \end{pmatrix},$$

$$n_7 = \begin{pmatrix} B_{11}h + O(h^2) \\ B_{02}h + O(h^2) \\ 1 + O(h^2) \end{pmatrix}.$$
The signed area enclosed by the spherical image of the path is approximately
\[ \pm \frac{1}{2} \sum_i = \| (n_i - n_0) \times (n_{i+2} - n_0) \| = 2(B_{20}B_{02} - B_{11}^2)h^2 + O(h^4). \]
where the index \( i \) is 1, 3, 5, and 7 taken in a cyclic manner, and where the sign is in the \( B_{20}B_{02} - B_{11}^2 \) factor. The expression on the left using formulas like (5.2) is the same when \( h \) is replaced by \( -h \), so the power series on the right must involve only even powers. Thus, the signed area of the spherical image is approximately
\[ \pm \frac{1}{2} \sum_i = \| (n_i - n_0) \times (n_{i+2} - n_0) \| = 2(B_{20}B_{02} - B_{11}^2)h^2 + O(h^4). \]
(5.3)

The quotient of the two areas (5.1) and (5.3) gives the curvature as
\[ B_{20}B_{02} - B_{11}^2 + O(h^2), \]
which is an \( O(h^2) \) approximation to the exact value in (2.7).

**Lemma 5.4.** For uniform data, the angle deficit method approximates the curvature to accuracy \( O(1) \).

**Proof.** Lemma 4.4 shows that the accuracy of the angle deficit method is at least \( O(1) \). With the four neighbouring points, \( P_1, P_3, P_5, \) and \( P_7 \) in Fig. 4, (4.4) and (4.5) give the curvature as \( \frac{1}{3}B_{20}B_{02} + O(h) \), which is the accuracy \( O(1) \). With eight points, \( P_1, P_2, \ldots, P_8 \) in Fig. 4, (4.4) and (4.5) give the curvature as \( \frac{1}{3}(B_{20}B_{02} - 2B_{11}^2) + O(h) \), which is also accuracy \( O(1) \). Notice that the combination of 1/3 (angle deficit with 4 points) +2/3 (angle deficit with 8 points) is \( B_{20}B_{02} - B_{11}^2 + O(h) \), which is an \( O(h) \) approximation to the curvature. Numerical results suggest that this combination is actually an \( O(h^2) \) approximation to the curvature.

**6. Numerical examples**

Numerical results are given to support the theoretical results in the previous sections. Let the surface containing the origin with formula
\[ z = -x + y + 2x^2 - xy - y^2 + x^3 - x^2y - xy^2 + 2y^3 \]
be \( S_1 \) (see Fig. 6). The unit normal and curvature at the origin are \( \frac{1}{\sqrt{3}}(1, -1, 1)^T \) and \(-1\).

Consider the eight non-uniform data points near the origin with \( x-y \) values \( P_1, P_2, \ldots, P_8 \) as shown in Fig. 5, where \( h \) is the adjustable spacing. The first, second, fourth, fifth, and seventh of the above points are used in the quadratic fit method. The error in the unit normal is calculated by finding the norm of the vector difference between the exact unit normal and the approximate unit normal. Table 1 shows the errors at the origin of various methods. It includes the errors of the unit normal and curvature from the quadratic fit method, the error in the unit normal from the average of normals method, the error in the curvature from the spherical image method (where \( O(h^2) \) approximations to the unit normals are calculated by extrapolation as described before Lemma 4.3), and the
Fig. 5. Eight neighbouring non-uniform data points of \( O, R(x, y) \) defined in (2.6).

Table 1

<table>
<thead>
<tr>
<th>( h )</th>
<th>Quadratic fit normal</th>
<th>Average of normals</th>
<th>Quadratic fit curvature</th>
<th>Spherical image curvature</th>
<th>Angle deficit curvature</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/16</td>
<td>5.889e–03</td>
<td>1.033e–02</td>
<td>–1.244e–01</td>
<td>–6.488e–02</td>
<td>–6.219e–02</td>
</tr>
<tr>
<td>1/32</td>
<td>1.464e–03</td>
<td>3.516e–03</td>
<td>–5.334e–02</td>
<td>–1.832e–02</td>
<td>–3.236e–02</td>
</tr>
<tr>
<td>1/64</td>
<td>3.656e–04</td>
<td>1.412e–03</td>
<td>–2.463e–02</td>
<td>–5.685e–03</td>
<td>–2.683e–02</td>
</tr>
<tr>
<td>1/128</td>
<td>9.136e–05</td>
<td>6.327e–04</td>
<td>–1.182e–02</td>
<td>–1.980e–03</td>
<td>–2.630e–02</td>
</tr>
</tbody>
</table>

The next five figures give information over the whole region \(-1 \leq x, y \leq 1\). The exact curvature of \( S_1 \) near the origin is shown in Fig. 7. Some imagination is required to see the correspondence of Figs. 6 and 7 since the \( x \)- and \( y \)-axes in the two figures are not aligned.

The data comes from a uniform grid in the \( x-y \) plane with \( h = 0.04 \) that has been perturbed by pseudo-random numbers. The errors in the quadratic fit approximations to the normal and curvature of \( S_1 \) are shown in Figs. 8 and 10. The errors in the average of normals approximation to the normal of \( S_1 \) are shown in Fig. 9. Exact normals were used with the spherical image method to give the errors in curvature of that method shown in Fig. 11.

Eight uniform data points near the origin are shown in Fig. 4. Table 2 shows the errors at the origin of various methods. It includes the errors of the normal and curvature from the finite difference formulas, the error in the unit normal from the average of normals method, the error in the curvature from the spherical image method, and the error in the curvature error in the curvature from the angle deficit method. The numerical results in Table 1 are compatible with the asymptotic results in Lemmas 4.1, 4.2, 4.3, and 4.4.
Fig. 6. Surface $S_1$ for $-1 \leq x, y \leq 1$.

Fig. 7. The curvature of surface $S_1$ for $-1 \leq x, y \leq 1$. 
Fig. 8. Error in the approximation to the normal by the quadratic fit method on $S_1$ with non-uniform data for $-1 \leq x, y \leq 1$.

Fig. 9. Error in the approximation to the normal by the average of normals method on $S_1$ with non-uniform data for $-1 \leq x, y \leq 1$. 
Fig. 10. The absolute value of the error in the approximation to the curvature by the quadratic fit method on $S_1$ with non-uniform data for $-1 \leq x, y \leq 1$.

Fig. 11. The absolute value of the error in the approximation to the curvature by the spherical image method on $S_1$ with non-uniform data for $-1 \leq x, y \leq 1$. 
Table 2
Errors for several \( h \) values of various methods that use uniform data from \( S_1 \)

<table>
<thead>
<tr>
<th>( h )</th>
<th>Finite diff. normal</th>
<th>Average of normals</th>
<th>Finite diff. curvature</th>
<th>Spherical image curvature</th>
<th>Angle deficit curvature</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/8</td>
<td>1.296e–02</td>
<td>2.144e–02</td>
<td>–6.466e–02</td>
<td>–3.644e–01</td>
<td>8.587e–02</td>
</tr>
<tr>
<td>1/16</td>
<td>3.202e–03</td>
<td>5.212e–03</td>
<td>–1.576e–02</td>
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<td>1.472e–01</td>
</tr>
<tr>
<td>1/32</td>
<td>7.981e–04</td>
<td>1.294e–03</td>
<td>–3.915e–03</td>
<td>–2.102e–02</td>
<td>1.618e–01</td>
</tr>
<tr>
<td>1/128</td>
<td>4.984e–05</td>
<td>8.068e–05</td>
<td>–2.442e–04</td>
<td>–1.303e–03</td>
<td>1.664e–01</td>
</tr>
</tbody>
</table>

Fig. 12. Errors in the approximation to the normal by the finite difference method on \( S_1 \) with uniform data for \(-1 \leq x, y \leq 1\).

from the angle deficit method. The numerical results in Table 2 are compatible with the asymptotic results in Lemmas 5.1, 5.2, 5.3, and 5.4.

The next four figures show errors in the whole region \(-1 \leq x, y \leq 1\). The data comes from a uniform grid in the \( x-y \) plane with \( h = 0.04 \). The errors in the finite difference approximations to the normal and curvature of \( S_1 \) are shown in Figs. 12 and 14. The errors in the average of normals approximation to the normal of \( S_1 \) are shown in Fig. 13. Exact normals were used with the spherical image method to give the errors in curvature shown in Fig. 15.
Fig. 13. Errors in the approximation to the normal by the average of normals method on $S_1$ with uniform data for $-1 \leq x, y \leq 1$.

Fig. 14. The absolute value of the errors in the approximation to the curvature by the finite difference method on $S_1$ with uniform data for $-1 \leq x, y \leq 1$. 
Fig. 15. The absolute value of the errors in the approximation to the curvature by the spherical image method on $S_1$ with uniform data for $-1 \leq x, y \leq 1$.

Table 3
Comparison of asymptotic errors for non-uniform data

<table>
<thead>
<tr>
<th>Method</th>
<th>Normal</th>
<th>Curvature</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quadratic fit</td>
<td>$O(h^2)$</td>
<td>$O(h)$</td>
</tr>
<tr>
<td>Average of normals</td>
<td>$O(h)$</td>
<td></td>
</tr>
<tr>
<td>Spherical image given $O(h^2)$ normals</td>
<td></td>
<td>$O(h)$</td>
</tr>
</tbody>
</table>

Table 4
Comparison of asymptotic errors for uniform data

<table>
<thead>
<tr>
<th>Method</th>
<th>Normal</th>
<th>Curvature</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finite difference</td>
<td>$O(h^2)$</td>
<td>$O(h^2)$</td>
</tr>
<tr>
<td>Average of normals</td>
<td>$O(h^2)$</td>
<td></td>
</tr>
<tr>
<td>Spherical image</td>
<td></td>
<td>$O(h^2)$</td>
</tr>
</tbody>
</table>

7. Conclusions

A brief summary of the asymptotic results in this paper appears in Tables 3 and 4. It must be mentioned that the asymptotic error indicates how a method will behave as $h$ tends zero. If $h$ is not close enough to zero, asymptotic results cannot properly be used to compare methods. In the approximation of curvatures for non-uniform data in Table 1,
the angle deficit method, whose error is $O(1)$, is numerically more accurate than the other methods for $h = 1/8$. As $h$ gets smaller, the other methods, whose errors are $O(h)$, become numerically more accurate than the angle deficit method. The asymptotic analysis does have several advantages over using numerical results as a way to compare approximations. An asymptotic analysis applies to a whole class of problems, while any numerical result is just one example. Asymptotic analysis is a mathematical method so it does not depend on computer programs being correct or on finite precision numbers behaving predictably. Both of the latter problems can render numerical results invalid without any indication of that fact. One expects the asymptotic results to show the behaviour of the various methods when the spacing between data points is small.

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